DELIGNE'S THEOREM ON TENSOR CATEGORIES

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Abstract. We look at and explain the concepts needed to understand the statement of Deligne's Theorem on Tensor Categories, which states that an arbitrary tensor category is equivalent to the category of representations of some super group. Throughout the paper we assume a background in only basic undergraduate mathematics.

Contents

1. Introduction

Supermathematics is the study of algebraic structures (vector spaces, algebras, modules, groups, etc.) that have a decomposition into "even" and "odd" pieces. Besides the purely mathematical reason to study supermath, one of the main motivations for supermath is the theory of supersymmetry. Supersymmetry is how supermath acquired such a "super" name and is where supermath offers much of its application and derives its motivation. Supersymmetry is a field within partical physics which studies the nature of fermions and bosons. An overly simplified example of a super vector space is if you consider two particles, either fermions or bosons, their spin can be represented as elements of a super vector space. When we take these two particles and allow them to interact, their resulting spin will be exactly the tensor product of the spins of the individual particles. This is just a simple example but supersymmetry works with complex theorems and results using supermath.

Deligne's Theorem on Tensor Categories was first published in 2002 and was a result drawn from Deligne's past work. This theorem ties together multiple fields of math and gives an equivalence to more easily work with tensor categories. The goal of this paper is actually quite simple, we want to be able to clearly understand and state Deligne's Theorem on Tensor Categories. To do so, we will start with simple concepts in super math, representation theory and category theory then build these up with clear examples. Then we will bring together the concepts and examples to fully understand Deligne's Theorem.

This paper is aimed at readers with at least a background of undergraduate level algebra. Specifically, we assume the reader has taken some courses on basic group theory and has seen many of the linear algebra concepts taught at the undergraduate level. Althought there are some concepts looked at in this paper that may be beyond the understanding of an undergraduate student, our hope is to properly explain and build up from known concepts so that our reader can conceptualize Delgine's Theorem.

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2. A Brief Introduction to Supermath

In this section, we look at the "super" analog of some of the mathematical structures that we already know and studied to certain extent. While doing so, we will develop the terms and examples that are common place while working in supermath. Both the terms and examples will be used throughout this paper as we build more advanced mathematical structures. The goal of this section is to give the reader a strong foundation in supermath so that we may build fluidly to our final theorem.

Definition 2.1 (Super Vector Space)**.** A *super vector space* (SVS) or *$\mathbb{Z}_2$-graded vector space* is a vector space, V, over a field, $\mathbb{K}$, with a direct sum decomposition, $V = V_0 \oplus V_1$.

When working with super vector spaces, we will view the subscripts of the subspaces as $0, 1 \in \mathbb{Z}_2$.

Definition 2.2 (Homogeneous)**.** Let V be a super vector space. An element of V is said to be *homogeneous* if it is an element of either V_0 or V_1.

Definition 2.3 (Parity)**.** The *parity* of nonzero homogeneous elements $v \in V$, denoted by $p(v) = |v|$, is defined as:

$$p(v) = |v| = \begin{cases} 0 & \text{if } v \in V_0, \\ 1 & \text{if } v \in V_1. \end{cases}$$

We call all elements of V_0 *even* and V_1 *odd*.

Definition 2.4 (Dimension of a SVS)**.** Given a finite-dimensional super vector space, $V = V_0 \oplus V_1$, where $\dim V_0 = p$ and $\dim V_1 = q$ then we say that V is of *dimension* $p|q$.

Definition 2.5 (Linear Transformation)**.** Let V and W be vector spaces over $\mathbb{K}$. A *linear transformation* from V to W is a map, $f : V \to W$, such that for all $x, y \in V$ and $k \in \mathbb{K}$ we have

$$f(x) + f(y) = f(x + y) \text{ and } f(kx) = kf(x).$$

We let $\mathscr{L}(V, W)$ denote the set of all linear transformations $V \to W$. Notice that $\mathscr{L}(V, W)$ is also a vector space under pointwise addition and scalar multiplication.

Definition 2.6 (Dual Space)**.** Let V be a vector space. We define the *dual space*, denoted V^*, to be the vector space $\mathscr{L}(V, \mathbb{K})$.

Remark 2.7. Notice that if V is of dimension p with basis $B = \{b_1, \ldots, b_p\}$, then V^* is also of dimension p with basis $B^\vee = \{b_i^\vee \mid b_i \in B\}$, where $b_i^\vee(b_j) = 1$ if and only if $b_i = b_j$. Furthermore, if V is a SVS then V^* is also a SVS where $p(b_i^\vee) = p(b_i)$.

Definition 2.8 (Grade-Preserving/Grade-Reversing)**.** A linear transformation, $f\colon V \to W$, between super vector spaces is said to be *grade-preserving* or *even* if $f(V_i) \subseteq W_i$ for $i = 0$ and 1. It is called *grade-reversing* or *odd* if $f(V_i) \subseteq W_{1-i}$.

Definition 2.9 (Transpose)**.** Given a linear map $f\colon V \to W$ then the *transpose map* is $f^*\colon W^* \to V^*$ defined for all $g \in W^*$ via

$$f^*(g) = g \circ f.$$

Notice that while working with SVS $p(f^*) = p(f)$.

Definition 2.10 (Isomorphic)**.** Two SVS, V and W, are *isomorphic* if there exists a grade-preserving bijective linear transformation between V and W. Such a map is called an *isomorphism*. If there exists such a transformation, then we write $V \cong W$.

Example 2.11. Consider the SVS $\mathbb{K}^{p|q} = \mathbb{K}^p \oplus \mathbb{K}^q$, for some field $\mathbb{K}$, where $\mathbb{K}_0^{p|q} = \mathbb{K}^p$ and $\mathbb{K}_1^{p|q} = \mathbb{K}^q$. Recall that any vector space V of dimension p is isomorphic to $\mathbb{K}^p$. Similarly, if V is a super vector space with $\dim V = p|q$ then $V \cong \mathbb{K}^{p|q}$.

Example 2.12. Let V and W be SVS. Notice $\mathscr{L}(V, W)$ itself is a super vector space with $\mathscr{L}(V, W) = \mathscr{L}(V, W)_0 \oplus \mathscr{L}(V, W)_1$ defined by:

$$\mathscr{L}(V, W)_0 = \{f\colon V \to W \mid f \text{ grade-preserving}\},$$
$$\mathscr{L}(V, W)_1 = \{f\colon V \to W \mid f \text{ grade-reversing}\}.$$

Example 2.13. Another example to consider is the vector space of $m \times n$ matrices, denoted $M_{m\times n}(\mathbb{K})$. We turn these matrices into super matrices by partitioning both the rows and colums into two. For example, the following $m \times n$ matrix can be turned into a super matrix by

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \to \left[\begin{array}{ccc|ccc} a_{1,1} & \cdots & a_{1,r} & a_{1,r+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{s,1} & \cdots & a_{s,r} & a_{s,r+1} & \cdots & a_{s,n} \\ \hline a_{s+1,1} & \cdots & a_{s+1,r} & a_{s+1,r+1} & \cdots & a_{s+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,r} & a_{m,r+1} & \cdots & a_{m,n} \end{array}\right] = \left[\begin{array}{c|c} X_{00} & X_{01} \\ \hline X_{10} & X_{11} \end{array}\right].$$

We say that these matrices are of dimension $s|p \times r|q$ where $s + p = m$ and $r + q = n$, denoted $M_{s|p\times r|q}(\mathbb{K})$. A matrix as above is defined to be even if $X_{01} = X_{10} = 0$, and odd if $X_{00} = X_{11} = 0$. Thus we have

$$M_{s|p\times r|q}(\mathbb{K}) = M_{s|p\times r|q}(\mathbb{K})_0 \oplus M_{s|p\times r|q}(\mathbb{K})_1.$$

Definition 2.14 (Bilinear Map)**.** A *bilinear map*, $B\colon V \times W \to U$, is a function such that for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $k \in \mathbb{K}$, we have

- $B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2)$,
- $B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$,
- $B(kv, w) = kB(v, w) = B(v, kw)$.

Definition 2.15 (Free Vector Space)**.** Given a set, S, the *free vector space* generated by S is

$$F(S) = \{\phi\colon S \to \mathbb{K} \mid \phi(s) = 0 \text{ for all except finitely many } s \in S\}.$$

With the usual pointwise addition and scalar multiplication of functions this is, in fact, a vector space. We will write $\sum_{s\in S} \phi(s)s$ instead of $\phi(s)$.

Definition 2.16 (Tensor Product)**.** Let V and W be vector spaces over $\mathbb{K}$. We define $Z \subseteq F(V \times W)$ to be the subspace defined by

$$Z = \left\{ \begin{array}{c} (v_1 + v_2, w) - (v_1, w) - (v_2, w), \\ (v, w_1 + w_2) - (v, w_1) - (v, w_2), \\ k(v, w) - (kv, w), \\ k(v, w) - (v, kw) \end{array} \middle| v, v_1, v_2 \in V, w, w_1, w_2 \in W, k \in \mathbb{K} \right\}.$$

The *tensor product* of V and W is the quotient space $F(V \times W)/Z$, denoted by $V \otimes W$. We write $v \otimes w$ for $(v, w) + Z$. The tensor product has the following properties for all $v, v_1, v_2 \in V,\ w, w_1, w_2 \in W$ and $k \in \mathbb{K}$:

$$\begin{aligned} (v_1 + v_2) \otimes w &= (v_1 \otimes w) + (v_2 \otimes w), \\ v \otimes (w_1 + w_2) &= (v \otimes w_1) + (v \otimes w_2), \text{ and} \\ kv \otimes w &= v \otimes kw. \end{aligned}$$

Lastly and foremost, $V \otimes W$ has a universal mapping property. This property states that for all vector spaces U and all bilinear maps $B\colon V \times W \to U$ there exists a unique linear map $\ell\colon V \otimes W \to U$ such that for all $v \in V$ and $w \in W$, $B(v, w) = \ell(v \otimes w)$.

Remark 2.17 (Dimension of $V \otimes W$)**.** Let V and W be finite-dimensional vector spaces with bases $b = \{b_1, \dots, b_p\}$ and $B = \{B_1, \dots, B_q\}$ respectively. The tensor product $V \otimes W$ has basis $b \otimes B = \{b_i \otimes B_j \mid b_i \in b \text{ and } B_j \in B\}$. Thus we know that if $\dim(V) = p$ and $\dim(W) = q$ then $\dim(V \otimes W) = pq$. For more on this refer to [1, §13].

Definition 2.18 (Tensor Product of Super Vector Spaces)**.** Given SVS $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$, the resulting $V \otimes W$ is naturally a super vector space with the $\mathbb{Z}_2$-grading

$$\begin{gathered} (V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1) \quad \text{and} \\ (V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0). \end{gathered}$$

Definition 2.19 (Superring)**.** A *superring* or *$\mathbb{Z}_2$-graded ring* is a ring, R, with the decompostion $R = R_0 \oplus R_1$ where the product map $R \times R \to R$ is even. As with SVS, the elements of R_0 are called *even* and elements of R_1 *odd*.

Definition 2.20 (Ring Homomorphism)**.** A *ring homomorphism* is an even map $f\colon R \to S$, with R and S superrings, such that for all $r, s \in R$

$$\begin{aligned} f(r+s) &= f(r) + f(s), \\ f(r \cdot s) &= f(r) \cdot f(s), \text{ and} \\ f(1_R) &= 1_S. \end{aligned}$$

Definition 2.21 (Discrete Supergroup)**.** A *discrete supergroup* is a $\mathbb{Z}_2$-graded group, $(G, \cdot)$, where $\cdot$ is such that $G_i \cdot G_j \subseteq G_{i+j}$. A $\mathbb{Z}_2$-graded group is such that $G = G_0 \sqcup G_1$, where $\sqcup$ is the disjoint union.

3. Superalgebras

In this section, we start by defining the "super" structure on *associative algebras* and give multiple examples of such structures. We then proceed by defining and giving examples of other super structures directly relating to *superalgebras*, like *super coalgebras* and *supermodules*. To follow, we will bring some of these definitions together to define one of the most important super structures looked at in this section, *a super bialgebra*.

Definition 3.1 (Superalgebra)**.** An *associative superalgebra* over some field $\mathbb{K}$ is a triple (A, ∇, η) where A is a SVS over $\mathbb{K}$, $\nabla\colon A \otimes A \to A$ is an even associative linear map and $\eta\colon \mathbb{K} \to A$. Furthermore, we require that the following diagrams commute.

$$\begin{tikzcd}
A & A\otimes A \arrow[l, "\nabla"'] \\
A\otimes A \arrow[u, "\nabla"] & A\otimes A\otimes A \arrow[l, "\mathrm{id}_A\otimes\nabla"] \arrow[u, "\nabla\otimes\mathrm{id}_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[r, "\nabla"] & A & A\otimes A \arrow[l, "\nabla"'] \\
& A \arrow[ul, "\eta\otimes\mathrm{id}_A"] \arrow[u, "\mathrm{id}_A"] \arrow[ur, "\mathrm{id}_A\otimes\eta"'] &
\end{tikzcd}$$

Notice when we say that $\mathrm{id}_A \otimes \eta\colon A \to A \otimes A$ the domain is actually $A \otimes \mathbb{K} \cong A$. For $a, b \in A$, we often write $a \cdot b$ or ab for $\nabla(a \otimes b)$.

Definition 3.2 (Supercommutative)**.** A superalgebra, A, is said to be *supercommutative* if for all homogeneous $a_1, a_2 \in A$, we have

$$a_1 a_2 = (-1)^{|a_1||a_2|} a_2 a_1.$$

Example 3.3. Recall from Example 2.12 that $\mathscr{L}(V, W)$ forms a super vector space. Now consider $\mathscr{L}(V, V)$, which is also denoted $\mathrm{End}(V)$ or $\mathfrak{gl}(V)$. We define the multiplication to be regular composition of maps. To confirm that the image of $\mathrm{End}(V)_i \otimes \mathrm{End}(V)_j$ is in $\mathrm{End}(V)_{i+j}$, take $f, g \in \mathrm{End}(V)$ such that $p(f) = i$, $p(g) = j$ and $v \in V$ such that $p(v) = k$ then $p((f \circ g)(v)) = p(f(g(v))) = i + j + k$ thus $p(f \circ g) = i + j$.

Example 3.4. Consider the super vector space of supermatrices of dimension $p|q \times p|q$ over some field $\mathbb{K}$, denoted $M_{p|q}(\mathbb{K})$. To make this a superalgebra, we use usual matrix multiplication and show that $M_{p|q}(\mathbb{K})_i \cdot M_{p|q}(\mathbb{K})_j \subseteq M_{p|q}(\mathbb{K})_{i+j}$, notice the following

- If both $X, Y \in M_{p|q}(\mathbb{K})_0$ then
$$X \cdot Y = \left[\begin{array}{c|c} X_{00}Y_{00} & 0 \\ \hline 0 & X_{11}Y_{11} \end{array}\right],$$
thus $X \cdot Y \in M_{p|q}(\mathbb{K})_0$.

- If $X \in M_{p|q}(\mathbb{K})_0$ and $Y \in M_{p|q}(\mathbb{K})_1$ then
$$X \cdot Y = \left[\begin{array}{c|c} 0 & X_{00}Y_{01} \\ \hline X_{11}Y_{10} & 0 \end{array}\right],$$
thus $X \cdot Y \in M_{p|q}(\mathbb{K})_1$.
- If X is $M_{p|q}(\mathbb{K})_1$ and Y is $M_{p|q}(\mathbb{K})_0$ then
$$X \cdot Y = \left[\begin{array}{c|c} 0 & X_{01}Y_{11} \\ \hline X_{10}Y_{00} & 0 \end{array}\right],$$
thus $X \cdot Y \in M_{p|q}(\mathbb{K})_1$.
- If $X, Y \in M_{p|q}(\mathbb{K})_1$ then
$$X \cdot Y = \left[\begin{array}{c|c} X_{01}Y_{10} & 0 \\ \hline 0 & X_{10}Y_{01} \end{array}\right],$$
thus $X \cdot Y \in M_{p|q}(\mathbb{K})_0$.

Thus $M_{p|q}(\mathbb{K})_i \cdot M_{p|q}(\mathbb{K})_j \subseteq M_{p|q}(\mathbb{K})_{i+j}$ as required.

Remark 3.5. Let V be a SVS of dimension $p|q$ over $\mathbb{K}$. We can pick bases for V_0 and V_1, $b = \{b_1, b_2, \ldots, b_p\}$ and $B = \{B_1, B_2, \ldots, B_q\}$ respectively. We can show that $\mathrm{End}(V) \cong M_{p|q}(\mathbb{K})$ by $F\colon \mathscr{L}(V,V) \to M_{p|q}(\mathbb{K})$ defined for any $T \in \mathfrak{gl}(V)$
$$F(T) = \left[\begin{array}{ccc|ccc} [T(b_1)]_b & \cdots & [T(b_p)]_b & [T(B_1)]_b & \cdots & [T(B_q)]_B \\ \hline [T(b_1)]_B & \cdots & [T(b_p)]_B & [T(B_1)]_B & \cdots & [T(B_q)]_B \end{array}\right] = \left[\begin{array}{c|c} X_{00} & X_{01} \\ \hline X_{10} & X_{11} \end{array}\right],$$
where, given $x \in V$, $[T(x)]_Y$ denotes the column coordinate vector of $T(x)$ with respect to the basis Y. Now notice that if T is an even transformation then this matrix will be a block diagonal matrix with $X_{01} = X_{10} = [0]$ and if T is an odd transformation then the matrix will be a block off-diagonal matrix with $X_{00} = X_{11} = [0]$. To show that this is, in fact, an isomorphism is left to reader.

Example 3.6 (Polynomial Superalgebra)**.** A *polynomial superalgebra* over $\mathbb{K}$ is defined to be the algebra over $\mathbb{K}$ generated by p even variables, X_i, and q odd variables, Y_j, with relations $X_iX_j = X_jX_i$, $X_iY_j = Y_jX_i$, and $Y_iY_j = -Y_jY_i$ for all i and j. This algebra is denoted by $\mathbb{K}[X_1, \ldots, X_p \mid Y_1, \ldots, Y_q]$. An element of $\mathbb{K}[X_1, \ldots, X_p \mid Y_1, \ldots, Y_q]$ is called a *monomial* if it is only a product of variables. Furthermore, we determine the parity of a monomial by the sum of variables parities in $\mathbb{Z}_2$.

Example 3.7 (Rank 1 Clifford Algebra)**.** The *rank 1 Clifford algebra* is a superalgebra with the underlying SVS $\mathbb{C}[x]/(x^2-1) = \mathbb{C} \oplus \mathbb{C}x$, where $\mathbb{C}$ is even and $\mathbb{C}x$ is odd, and multiplication being the standard polynomial multiplication. With this multiplication it is easy to verify that $(\mathbb{C}[x]/(x^2-1))_i \cdot (\mathbb{C}[x]/(x^2-1))_j \subseteq (\mathbb{C}[x]/(x^2-1))_{i+j}$ is true for all i and j.

Definition 3.8 (Tensor of Algebras)**.** Given A and B algebras then naturally $A \otimes B$ is a vector space. To define an algebra structure on $A \otimes B$, we define multiplication of any $a_1 \otimes b_1, a_2 \otimes b_2 \in A \otimes B$ by
$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2.$$

Definition 3.9 (Tensor of Superalgebras)**.** Analogously if A and B are both superalgebras then $A \otimes B$ is a SVS as seen in Definition 2.18. Furthermore, $A \otimes B$ will also be an superalgebra with multiplication defined, for all simple tensors $a_1 \otimes b_1, a_2 \otimes b_2 \in A \otimes B$, by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|a_2|\cdot|b_1|}(a_1a_2 \otimes b_1b_2).$$

Example 3.10 (Rank r Clifford Algebra)**.** Using the rank 1 Clifford algebra from Example 3.7 we can construct a *rank r Clifford algebra* by forming the tensor product $\mathbb{C}[x_1]/(x_1^2-1)\otimes \mathbb{C}[x_2]/(x_2^2-1)\otimes\cdots\otimes\mathbb{C}[x_r]/(x_r^2-1)$ with multiplication as defined in Definition 3.9. While working with rank r Clifford Algebras we write x_i for $1\otimes\cdots\otimes x_i\otimes\cdots\otimes 1$ for all $i \in \{1,\dots,r\}$. The rank r Clifford algebra has a basis given by monomials in the x_i. The parity 0 component is spanned by monomials of even degree and the parity 1 component is spanned by monomials of odd degree. Notice that the rank r Clifford algebra is a supercommutative superalgebra since for all $i, j \in \{1,\dots,r\}$ with $i \neq j$ we have $x_ix_j = (-1)x_jx_i$.

Definition 3.11 (Supermodule)**.** Let (A, ∇, η) be a superalgebra. A *right supermodule* or *right $\mathbb{Z}_2$-graded module*, M, is a SVS together with an even linear map $\sigma\colon M \otimes A \to M$, called an *action*, such that the following diagrams commute.

$$\begin{tikzcd}
M\otimes A \arrow[r, "\sigma"] & M & M\otimes A \arrow[l, "\sigma"'] \\
& M\otimes A\otimes A \arrow[ul, "\mathrm{id}_M\otimes\nabla"] \arrow[ur, "\sigma\otimes\mathrm{id}_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M & M\otimes A \arrow[l, "\sigma"'] \\
M \arrow[u, equal, "\mathrm{id}_M"] \arrow[ur, "\mathrm{id}_M\otimes\eta"'] &
\end{tikzcd}$$

These diagrams commute when we identify $M \cong M \otimes \mathbb{K}$. A *left supermodule* is defined analogously.

Example 3.12. Notice that $\mathbb{K}^{p|q} = \mathbb{K}^p \oplus \mathbb{K}^q$ over $M_{p|q}(\mathbb{K})$ with standard matrix multiplication forms a right supermodule (or a left supermodule if we view $\mathbb{K}^{p|q}$ as column vectors). To insure that this is a module, notice that $\mathbb{K}^{p|q}$ is an abelian group under standard vector addition, $1 \in M_{p|q}(\mathbb{K})$ is just the identity matrix and distributivity is inherited from distributivity of matrix multiplication over addition. Lastly to insure that $M_{p|q}(\mathbb{K})_i \cdot \mathbb{K}^{p|q}_j \subseteq \mathbb{K}^{p|q}_{i+j}$, consider the following cases:

- If $Y \in M_{p|q}(\mathbb{K})_0$ and $X \in \mathbb{K}^{p|q}_0$ then
$$Y \cdot X = \left[\begin{array}{c|c} Y_{00} & 0 \\ \hline 0 & Y_{11} \end{array}\right] \cdot \left[\begin{array}{c} X_0 \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} Y_{00}X_0 \\ \hline 0 \end{array}\right] \in \mathbb{K}^{p|q}_0.$$
- If $Y \in M_{p|q}(\mathbb{K})_0$ and $X \in \mathbb{K}^{p|q}_1$ then
$$Y \cdot X = \left[\begin{array}{c|c} Y_{00} & 0 \\ \hline 0 & Y_{11} \end{array}\right] \cdot \left[\begin{array}{c} 0 \\ \hline X_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline Y_{11}X_1 \end{array}\right] \in \mathbb{K}^{p|q}_1.$$
- If $Y \in M_{p|q}(\mathbb{K})_1$ and $X \in \mathbb{K}^{p|q}_0$ then
$$Y \cdot X = \left[\begin{array}{c|c} 0 & Y_{01} \\ \hline Y_{10} & 0 \end{array}\right] \cdot \left[\begin{array}{c} X_0 \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline Y_{10}X_0 \end{array}\right] \in \mathbb{K}^{p|q}_1.$$
- If $Y \in M_{p|q}(\mathbb{K})_1$ and $X \in \mathbb{K}^{p|q}_1$ then
$$Y \cdot X = \left[\begin{array}{c|c} 0 & Y_{01} \\ \hline Y_{10} & 0 \end{array}\right] \cdot \left[\begin{array}{c} 0 \\ \hline X_1 \end{array}\right] = \left[\begin{array}{c} Y_{01}X_1 \\ \hline 0 \end{array}\right] \in \mathbb{K}^{p|q}_0.$$

Concluding that in fact $M_{p|q}(\mathbb{K})_i \cdot \mathbb{K}^{p|q}_j \subseteq \mathbb{K}^{p|q}_{i+j}$ for all $i, j \in \mathbb{Z}_2$, as required.

Definition 3.13 (Super Coalgebra)**.** A *super coalgebra* is a triple (C, Δ, ϵ) where C is a SVS over $\mathbb{K}$ and $\Delta\colon C \to C \otimes C$ and $\epsilon\colon C \to \mathbb{K}$ are even morphisms such that the following diagrams commute.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes \mathrm{id}_C"] \\ C \otimes C \arrow[r, "\mathrm{id}_C \otimes \Delta"] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \otimes C \arrow[r, "\mathrm{id}_C \otimes \epsilon"] & C & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{id}_C"'] \\ & C \arrow[ul, "\Delta"] \arrow[u, "\mathrm{id}_C"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

That is, for all $c \in C$, $(\mathrm{id}_C \otimes \epsilon) \circ \Delta(c) = c = (\epsilon \otimes \mathrm{id}_C) \circ \Delta(c)$. We call Δ the *comultiplication* and ϵ the *counit*.

Remark 3.14. While working with coalgebras, we write $\Delta(c) = \sum c_1 \otimes c_2$ and we will sometimes write $c = c_1\epsilon(c_2) = \epsilon(c_1)c_2$ instead of $(\mathrm{id}_C \otimes \epsilon) \circ \Delta(c) = c = (\epsilon \otimes \mathrm{id}_C) \circ \Delta(c)$.

Definition 3.15 (Super Bialgebra)**.** A *super bialgebra* is a 5-tuple $(B, \nabla, \eta, \Delta, \epsilon)$ such that (B, ∇, η) is an associative superalgebra over some field $\mathbb{K}$, (B, Δ, ϵ) is a super coalgebra over $\mathbb{K}$, and the following diagrams are commutative.

$$\begin{tikzcd} B \otimes B \arrow[r, "\nabla"] \arrow[d, "\Delta \otimes \Delta"] & B \arrow[r, "\Delta"] & B \otimes B \\ B \otimes B \otimes B \otimes B \arrow[rr, "\mathrm{id}_B \otimes \mu \otimes \mathrm{id}_B"] & & B \otimes B \otimes B \otimes B \arrow[u, "\nabla \otimes \nabla"'] \end{tikzcd}$$

In the above, $\mu\colon B \otimes B \to B \otimes B$ is the flip operator.

$$\begin{tikzcd} B \otimes B \arrow[rr, "\nabla"] \arrow[dr, "\epsilon \otimes \epsilon"'] & & B \arrow[dl, "\epsilon"] \\ & \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K} & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{K} \otimes K \cong \mathbb{K} \arrow[dl, "\eta \otimes \eta"'] \arrow[dr, "\eta"] & \\ B \otimes B & & B \arrow[ll, "\Delta"] \end{tikzcd}$$

$$\begin{tikzcd} & B \arrow[dr, "\epsilon"] & \\ \mathbb{K} \arrow[ur, "\eta"] \arrow[rr, equal, "\mathrm{id}_{\mathbb{K}}"] & & \mathbb{K} \end{tikzcd}$$

Example 3.16. Recall the polynomial superalgebra, $\mathbb{K}[X_1, \dots, X_p \mid Y_1, \dots, Y_q]$, as described in Example 3.6. We make this into a super bialgebra by defining

$$\epsilon(f) = f(0, \dots, 0 \mid 0, \dots, 0)$$

for all $f \in \mathbb{K}[X_1, \dots, X_p \mid Y_1, \dots, Y_q]$ and we define

$$\Delta(W) = W \otimes 1 + 1 \otimes W$$

where $W \in \{X_1, \dots, X_p, Y_1, \dots, Y_q\}$ and extend by linearity and multiplicatively. It is important to notice that the relation $Y_iY_j = -Y_jY_i$ of the polynomial superalgebra implies that $Y_i^2 = 0$. Thus for all $Y_i \in \{Y_1, \dots, Y_q\}$, $\Delta(Y_i^2) = 0$. To verfiy this consider the following.

$$\begin{aligned} \Delta(Y_i^2) &= \Delta(Y_i)\Delta(Y_i) = (Y_i \otimes 1 + 1 \otimes Y_i)^2 \\ &= Y_i^2 \otimes 1 + Y_i \otimes Y_i - Y_i \otimes Y_i + 1 \otimes Y_i^2 = 0 \end{aligned}$$

4. Super Hopf Algebras

In this section, we take the super bialgebras previously defined and use them to define *super Hopf algebras*. Continuing, we start by first looking at some examples of both $\mathbb{Z}_2$-graded Hopf algebras and ungraded Hopf algebra. One example to pay special attention to is the *group algebra* example, since it will be the motivation for when we talk about supergroups. Lastly, we will look at and prove some important properties concerning the duals of superalgebras, super coalgebras, super bialgebras and super Hopf algebras.

Definition 4.1 (Super Hopf Algebra)**.** A *super Hopf algebra* is a 6-tuple $(H, \nabla, \eta, \Delta, \epsilon, S)$ where $(H, \nabla, \eta, \Delta, \epsilon)$ is a super bialgebra and S is an even linear map $S\colon H \to H$, called *the antipode*, such that the following diagram commutes.

$$\begin{tikzcd} H \otimes H \arrow[r, "\nabla"] & H & H \otimes H \arrow[l, "\nabla"'] \\ H \otimes H \arrow[u, "\mathrm{id}_H \otimes S"] & H \arrow[l, "\Delta"] \arrow[r, "\Delta"'] \arrow[u, "\eta\circ\epsilon"] & H \otimes H \arrow[u, "S\otimes \mathrm{id}_H"'] \end{tikzcd} \tag{4.1}$$

That is for all $h \in H$

$$\nabla \circ (\mathrm{id}_H \otimes S) \circ \Delta(h) = \eta \circ \epsilon(h) = \nabla \circ (S \otimes \mathrm{id}_H) \circ \Delta(h).$$

Example 4.2. Continuing with the polynomial superalgebra from Example 3.16, we will define the antipode, S, of $H = \mathbb{K}[X_1, \ldots, X_p \mid Y_1, \ldots, Y_q]$ so that it is a super Hopf algebra. We define $S\colon H \to H$ to be the unique algebra homomorphism such that for all $h \in \{X_1, \ldots, X_p, Y_1, \ldots, Y_q\}$ we have

$$S(h) = -h.$$

Now to verify that this antipode, S, satisfies the diagram in Definition 4.1, we can look at $h \in \{X_1, \ldots, X_p, Y_1, \ldots, Y_q\}$ and consider

$$\begin{aligned} \nabla(\mathrm{id}_H \otimes S(\Delta(h))) &= \nabla(\mathrm{id}_H \otimes S(h \otimes 1 + 1 \otimes h)) \\ &= \nabla(h \otimes 1 - 1 \otimes h) = h - h = 0, \\ \eta(\epsilon(h)) &= \eta(0) = 0, \\ \nabla(S \otimes \mathrm{id}_H(\Delta(h))) &= \nabla(S \otimes \mathrm{id}_H(h \otimes 1 + 1 \otimes h)) \\ &= \nabla(-h \otimes 1 + 1 \otimes h) = -h + h = 0. \end{aligned}$$

Thus for all $h \in \{X_1, \ldots, X_p, Y_1, \ldots, Y_q\}$, we have $\nabla(\mathrm{id}_H \otimes S(\Delta(h))) = \eta(\epsilon(h)) = \nabla(S \otimes \mathrm{id}_H(\Delta(h)))$. Since $\{X_1, \ldots, X_p, Y_1, \ldots, Y_q\}$ generates H, it follows that (4.1) commutes. We conclude that H is a super Hopf algebra.

Example 4.3 (Tensor Algebra)**.** Let V be a SVS over $\mathbb{K}$. Thus $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ is a SVS where $V^{\otimes 0} = \mathbb{K}$ and the parity decomposition is as described in Definition 2.18. We call $T(V)$ a *tensor algebra*. It will be a superalgebra where $\nabla\colon T(V) \otimes T(V) \to T(V)$ is defined for all $\bigotimes_{i=1}^{k} v_i, \bigotimes_{j=1}^{\ell} w_j \in T(V)$ by

$$\nabla\left(\left(\bigotimes_{i=1}^{k} v_i\right) \otimes \left(\bigotimes_{j=1}^{\ell} w_j\right)\right) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell,$$

and extend by linearity. Continuing, we let $\Delta\colon T(V) \to T(V) \otimes T(V)$, $\epsilon\colon T(V) \to \mathbb{K}$ and $S\colon T(V) \to T(V)$ be algebra homomorphisms. Now we define these maps such that $T(V)$ is a super Hopf algebra. For all $x \in V$ we define

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x, \ \Delta(1) = 1 \otimes 1, \\ \epsilon(x) &= 0, \ \text{and} \\ S(x) &= -x, \end{aligned}$$

and extend linearly and multiplicatively to higher tensor powers. With these definitions it is straightforward to verify that $T(V)$ is a Hopf superalgebra.

Example 4.4 (Symmetric Algebra)**.** Consider the Hopf superalgebra, $T(V)$, from Example 4.3. We call the quotient algebra $T(V)/I$ the *symmetric algebra*, denoted $S(V)$, where I is the ideal generated by $\{x \otimes y - (-1)^{p(x)p(y)} y \otimes x \mid x, y \in V\}$. Now $S(V)$ is inheritely an algebra, Furthermore, the maps Δ, ϵ, ∇, η, and S as defined in Example 4.3 will induce maps on the quotient such that $S(V)$ will also be a super Hopf algebra. Notice that $S(V)$ is supercommutative since for all $x, y \in V$

$$0 = x \otimes y - (-1)^{p(x)p(y)} y \otimes x \implies x \otimes y = (-1)^{p(x)p(y)} y \otimes x.$$

Example 4.5 (Group Algebra)**.** Given a group G. The *group algebra* of G is the vector space

$$\mathbb{K}[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in \mathbb{K} \ \forall g \in G \right\},$$

with multiplication defined for all $a_g, b_h \in \mathbb{K}$ and $g, h \in G$ to be

$$(a_g g) \cdot (b_h h) = (a_g b_h) gh$$

and extended by linearity. Now to make this an example of a Hopf algebra, we define $\Delta\colon \mathbb{K}[G] \to \mathbb{K}[G] \otimes \mathbb{K}[G]$, $\epsilon\colon \mathbb{K}[G] \to \mathbb{K}$, and $S\colon \mathbb{K}[G] \to \mathbb{K}[G]$ for all $g \in G$ to be

$$\begin{aligned} \Delta(g) &= g \otimes 1 + 1 \otimes g, \\ \epsilon(g) &= \begin{cases} 1 \text{ if } g = e_G, \\ 0 \text{ if } g \neq e_G, \end{cases} \\ S(g) &= g^{-1}. \end{aligned}$$

Given these definitions, it is straightforward to verify that these maps satisfy the conditions of Definition 4.1. Thus $\mathbb{K}[G]$ is a Hopf algebra.

Throughout the rest of this work, we will need some commonplace identifications and the following lemma will provide us with the proofs of such identifications.

Lemma 4.6.
(i) If A is finite-dimensional vector space, then $\varphi\colon (A^* \otimes A^*) \to (A \otimes A)^*$ defined for all $f \otimes g \in A^* \otimes A^*$ and $a \otimes b \in A \otimes A$ via $\varphi(f \otimes g)(a \otimes b) = f(a)g(b)$ is an isomorphism when extended by linearity.

(ii) Similarly, the map $\psi\colon \mathbb{K}^* \to \mathbb{K}$ defined for all $f \in \mathbb{K}^*$ via $\psi(f) = f(1)$ is an isomorphism.

Proof. To prove (i), assume that A is of dimension p with basis $B = \{a_1, \ldots, a_p\}$. Thus, we know then that both $A^* \otimes A^*$ and $(A \otimes A)^*$ are of dimension p^2 with respective basis $\{a_i^* \otimes a_j^* \mid a_i, a_j \in B\}$ and $\{(a_i \otimes a_j)^* \mid a_i, a_j \in B\}$. To show that φ is an isomorphism, notice that for any $a_i^* \otimes a_j^* \in \{a_i^* \otimes a_j^* \mid a_i, a_j \in B\}$ and for all $a \otimes b \in A \otimes A$ we have $\varphi(a_i^* \otimes a_j^*)(a \otimes b) = (a_i \otimes a_j)^*(a \otimes b)$. Thus $\varphi(a_i^* \otimes a_j^*) = (a_i \otimes a_j)^*$ and we can easily conclude that φ is surjective and therefore an isomorphism.
The proof for (ii) is straightforward and follows immediately from the definition of ψ. ■

We will view the isomorphisms of Lemma 4.6 as identifications.

Proposition 4.7.
(1) Given a finite-dimensional superalgebra, (A, ∇, η), then the dual of said superalgebra, (A^*, ∇^*, η^*), is a super coalgebra. Furthermore, given a finite-dimensional super coalgebra (C, Δ, ϵ) then the dual, $(C^*, \Delta^*, \epsilon^*)$, is a superalgebra.

(2) If $(B, \nabla, \eta, \Delta, \epsilon)$ is a finite-dimensional super bialgebra then the dual $(B^*, \Delta^*, \epsilon^*, \nabla^*, \eta^*)$ is a super bialgebra.

(3) Given a finite-dimensional super Hopf algebra, $(H, \nabla, \eta, \Delta, \epsilon, S)$, then the dual of said Hopf algebra, $(H^*, \Delta^*, \epsilon^*, \nabla^*, \eta^*, S^*)$, is a super Hopf algebra.

Proof. To prove (1), take (A, ∇, η) to be a finite-dimensional superalgebra. We already know that the dual, A^*, is a SVS. To check that ∇^* and η^* satisfy the diagrams in Definition 3.13, notice that taking the dual of the A and the transposes of ∇ and η results in the following commutative diagrams.

$$\begin{tikzcd}
A^* \arrow[r, "\nabla^*"] \arrow[d, "\nabla^*"'] & A^* \otimes A^* \arrow[d, "\nabla^* \otimes \mathrm{id}_A^*"] \\
A^* \otimes A^* \arrow[r, "\mathrm{id}_{A^*} \otimes \nabla^*"] & A^* \otimes A^* \otimes A^*
\end{tikzcd}
\qquad
\begin{tikzcd}
A^* \otimes A^* \arrow[r, "\mathrm{id}_A^* \otimes \eta^*"] & A^* & A^* \otimes A^* \arrow[l, "\eta^* \otimes \mathrm{id}_A^*"'] \\
& A^* \arrow[ul, "\nabla^*"] \arrow[u, "\mathrm{id}_{A^*}"] \arrow[ur, "\nabla^*"'] &
\end{tikzcd}$$

Thus (A^*, ∇^*, η^*) is a super coalgebra.

Given a super coalgebra, (C, Δ, ϵ), we can use a similar process as above to check that $(C^*, \Delta^*, \epsilon^*)$ satisfies the axioms of a superalgebra. This is left to the reader.

To prove (2), take $(B, \nabla, \eta, \Delta, \epsilon)$ to be a finite-dimensional super bialgebra. Notice that (B, ∇, η) is an associative super algebra and by (1) we can conclude that (B^*, ∇^*, η^*) is a super coalgebra. Similarly, (B, Δ, ϵ) is a super coalgebra and again by (1) we then know that $(B^*, \Delta^*, \epsilon^*)$ is a super algebra. To verify that the diagrams in Definition 3.15 commute, notice that taking the dual of B and taking the tranposes of the maps $\nabla, \eta, \Delta,$ and ϵ yields the following commutative diagrams.

$$\begin{tikzcd}
B^* \otimes B^* \arrow[r, "\Delta^*"] \arrow[d, "\nabla^* \otimes \nabla^*"] & B^* \arrow[r, "\nabla^*"] & B^* \otimes B^* \\
B^* \otimes B^* \otimes B^* \otimes B^* \arrow[rr, "\mathrm{id}_{B^*} \otimes \mu \otimes \mathrm{id}_{B^*}"] & & B^* \otimes B^* \otimes B^* \otimes B^* \arrow[u, "\Delta^* \otimes \Delta^*"']
\end{tikzcd}$$

$$\begin{tikzcd} B^* \otimes B^* \arrow[rr, "\Delta^*"] \arrow[dr, "\eta^*\otimes\eta^*"'] & & B^* \arrow[dl, "\eta^*"] \\ & \mathbb{K}\otimes\mathbb{K} \cong \mathbb{K} & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{K}\otimes K \cong \mathbb{K} \arrow[dl, "\epsilon^*\otimes\epsilon^*"'] \arrow[dr, "\epsilon^*"] & \\ B^* \otimes B^* & & B^* \arrow[ll, "\nabla^*"] \end{tikzcd}$$

$$\begin{tikzcd} & B^* \arrow[dr, "\eta^*"] & \\ \mathbb{K} \arrow[ur, "\epsilon^*"] \arrow[rr, equal, "\mathrm{id}_{\mathbb{K}}"] & & \mathbb{K} \end{tikzcd}$$

Thus, it follows that $(B^*, \Delta^*, \epsilon^*, \nabla^*, \eta^*)$ is a super bialgebra.

To prove (3), let $(H, \nabla, \eta, \Delta, \epsilon, S)$ be a finite-dimensional super Hopf algebra. Notice that $(H, \nabla, \eta, \Delta, \epsilon)$ is a super bilagebra thus by (2) we know that $(H^*, \Delta^*, \epsilon^*, \nabla^*, \eta^*)$ is a super bilaglebra. Given this we need only verify that the antipode, $S^* \colon H^* \to H^*$, satisfies Diagram 4.1. Notice that when we consider the dual of H and the transposes of $\nabla, \eta, \Delta, \epsilon$, and S, we get the following diagram.

$$\begin{tikzcd} H^* \otimes H^* \arrow[r, "\Delta^*"] & H^* & H^* \otimes H^* \arrow[l, "\Delta^*"'] \\ H^* \otimes H^* \arrow[u, "\mathrm{id}_{H^*}\otimes S^*"] & H^* \arrow[l, "\nabla^*"] \arrow[r, "\nabla^*"'] \arrow[u, "\epsilon^*\circ\eta^*"] & H^* \otimes H^* \arrow[u, "S\otimes \mathrm{id}_{H^*}"] \end{tikzcd}$$

Thus $(H^*, \Delta^*, \epsilon^*, \nabla^*, \eta^*, S^*)$ is a super Hopf algebra. ■

Example 4.8. Recall the superalgebra, from Example 3.4, consisting of supermatrices. We use Proposition 4.7 to construct the super coalgebra, $C = M_{p|q}(\mathbb{K})^*$. We know C has a basis $f_{i,j}$ where for all $X \in M_{p|q}(\mathbb{K})$, $f_{i,j}(X)$ is defined as the (i, j)-th entry of X. In fact C is a SVS, where we define $f_{i,j}$ to be even if $1 \le i, j \le p$ or $p+1 \le i, j \le q$ and otherwise odd. To make this into a super coalgebra we know $\nabla^* \colon C \to C \otimes C$ and $\eta^* \colon C \to \mathbb{K}$ for all $f_{i,j} \in C$ by

$$\begin{aligned} \nabla^*(f_{i,j}) &= f_{i,j} \circ \nabla \text{ and} \\ \eta^*(f_{i,j}) &= f_{i,j} \circ \eta. \end{aligned}$$

Since we have specific definitions for ∇ and η, we can deduce specific equations for both ∇^* and η^*. First for ∇^*, consider for any $f_{i,j} \in C$ and $M, N \in M_{p|q}(\mathbb{K})$

$$\begin{aligned} \nabla^*(f_{i,j})(M \otimes N) = f_{i,j}(MN) &= \sum_{k=1}^{p+q} M_{i,k}N_{k,j} \\ &= \sum_{k=1}^{p+q} f_{i,k}(M)f_{k,j}(N) \\ &= \sum_{k=1}^{p+q} (f_{i,k} \otimes f_{k,j})(M \otimes N). \end{aligned}$$

To deduce η^*, first $\eta\colon \mathbb{K} \to M_{p|q}(\mathbb{K})$ is defined for all $k \in \mathbb{K}$ as $\eta(k) = k \cdot I$, where I is the identity matrix. Recall the identification of $\mathbb{K}^*$ with $\mathbb{K}$ given in Lemma 4.6. We compute:

$$\eta^*(f_{i,j}) = f_{i,j} \circ \eta(1) = f_{i,j}(I) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Now to verify that this satisfies the equation $\mathrm{id}_C \otimes \eta^*(\nabla^*(c)) = c = \eta^* \otimes \mathrm{id}_C(\nabla^*(c))$ for all $c \in C$, for all $f_{i,j} \in C$, consider:

$$\begin{aligned} \eta^* \otimes \mathrm{id}_C(\nabla^*(f_{i,j})) &= \eta^* \otimes \mathrm{id}_C \left(\sum_{k=1}^{p+q} f_{i,k} \otimes f_{k,j} \right) \\ &= \sum_{k=1}^{p+q} \eta^*(f_{i,k}) \otimes f_{k,j} = f_{i,j} = \sum_{k=1}^{p+q} f_{i,k} \otimes \eta^*(f_{k,j}) = \eta^* \otimes \mathrm{id}_C(\nabla^*(f_{i,j})). \end{aligned}$$

Thus C is a super coalgebra.

Example 4.9. Given Proposition 4.7, let us consider the dual of the Hopf algebra from Example 4.5, denoted by $\mathcal{O}(G)$. Notice that G generates the basis for $\mathbb{K}[G]$ thus the basis for $\mathbb{K}[G]^*$ is generated by maps from G to $\mathbb{K}$ which is the vector space, $\mathcal{O}(G)$. By Proposition 4.7, we compute S^*, ϵ^*, and η^* for all $f \in \mathcal{O}(G)$, $x \in G$, and $k \in \mathbb{K}$

$$\begin{aligned} S^*(f)(x) &= f \circ S(x) = f(x^{-1}), \\ \eta^*(f) &= f \circ \eta = f(e_G), \\ \epsilon^*(k) &= k \cdot 1_{\mathcal{O}(G)}. \end{aligned}$$

Now before explicitly computing Δ^* and ∇^*, it is important to notice that since G is finite then the map $\psi\colon \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G \times G)$ defined for all $f, g \in \mathcal{O}(G)$ and $(a, b) \in G \times G$ via $\psi(f \otimes g)(a, b) = f(a)g(b)$ is an isomorphism. With such an identification, we can deduce for all $f, g \in \mathcal{O}(G)$

$$\begin{aligned} \nabla^*(f) &= f \circ \nabla \text{ and} \\ \Delta^*(f \otimes g) &= (f \otimes g) \circ \Delta. \end{aligned}$$

Definition 4.10 (Cocommutatitive)**.** A coalgebra, (C, Δ, ϵ), is called *cocommutative* if the following diagram is commutative.

$$\begin{tikzcd} C \otimes C & C \arrow[l, "\Delta"'] \arrow[dl, "\Delta"] \\ C \otimes C \arrow[u, "\mu"] & \end{tikzcd}$$

Where $\mu\colon A \otimes B \to B \otimes A$ is the flip operator defined for all $a \in A$ and $b \in B$ via $\mu(a \otimes b) = b \otimes a$, and extended by linearity.

Example 4.11. Recall the Hopf algebra $\mathbb{K}[G]$ from Example 4.5. Notice that $\mathbb{K}[G]$ is cocommutative since for all $g \in \mathbb{K}[G]$, $\Delta(g) = 1 \otimes g + g \otimes 1 = \mu(\Delta(g))$.

5. Representation Theory

In this section, we briefly define *representations* of a group and some associated structures utilizing these representations. Although this section is short, it holds some important information that we will use later with our more complex supergroup structure.

Definition 5.1 (Representation)**.** A *representation* of a group G on a vector space V over $\mathbb{K}$ is a group homomorphism from G to $GL(V) = \{f \in \mathscr{L}(V,V) \mid f \text{ is invertible}\}$. That is, $\rho\colon G \to GL(V)$ is a map where $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$ for all $g_1, g_2 \in G$. We will call V a representation of G and leave the map ρ implied. We will also write $g \cdot v$ for $\rho(g)(v)$.

Definition 5.2 (G-module)**.** Equivalent to the notion of representations we define G-modules as followed. Let G be a group. A *left G-module* is a vector space, A, with a map $\rho\colon G \times A \to A$. The map ρ is defined for all $g, g_1, g_2 \in G$ and $a, a_1, a_2 \in A$ via

$$g \cdot (a_1 + a_2) = g \cdot a_1 + g \cdot a_2,$$
$$g_1 \cdot (g_2 \cdot m) = (g_1g_2) \cdot m.$$

We write $g \cdot a$ for $\rho(g, a)$.

Definition 5.3 (Subrepresentation)**.** Given a representation, ρ, of G on V then a subspace $W \subseteq V$ is called *G-invariant* if $\rho(g, w) \in W$ for all $g \in G$ and $w \in W$. A *subrepresentation* is the restriction of ρ to a G-invariant subspace, W.

We say $\rho\colon G \to GL(V)$ is *irreducible* if the only G-invariant subspaces are the zero subspace and the whole vector space V.

Definition 5.4 (Direct Sum of Representations)**.** Let G be a group and V and W two representations of G. The *direct sum of group representations* is the direct sum of vector spaces $V \oplus W$ with the action of $g \in G$ for all $(v, w) \in V \oplus W$ given by $g \cdot (v, w) = (g \cdot v, g \cdot w)$.

Definition 5.5 (Tensor Product of Representations)**.** Let (V, ρ) and (W, σ) be representations of some group, G. The *tensor product* $V \otimes W$ is a representation of G whose action, $\rho \otimes \sigma$, is defined for all $g \in G$ and $\sum v_i \otimes w_i \in V \otimes W$ by

$$(\rho \otimes \sigma)(g)(v \otimes w) = \rho(g)(v) \otimes \sigma(g)(w).$$

Example 5.6. Let V be a vector space. Considering the group of permutations on $X = \{1, \dots, n\}$, denoted S_n, we define the representation of S_n on $V^{\otimes n}$ by the action, ρ. We define ρ for all $\sigma \in S_n$ and all $v_1 \otimes \cdots \otimes v_n \in V_1 \otimes \cdots \otimes V_n$ as $\rho(\sigma, v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ and extend by linearity.

Definition 5.7 (Invariant)**.** Let (V, ρ) be a representation of G. We say that $v \in V$ is *invariant* if $\rho(g, v) = v$ for all $g \in G$.

Definition 5.8 (Partition)**.** Given $n \in \mathbb{N}$, a *partition of* n is a list of non-increasing natural numbers whose sum is n.

Example 5.9. As an example, all the possible partitions of 4 are $(1,1,1,1)$, $(2,1,1)$, $(2,2)$, $(3,1)$, and (4).

Remark 5.10. Later it will be important to know the fact that the irreducible representations of S_n are naturally enumerated by the partitions of n. We let V_λ denote the irreducible representation of S_n corresponding the the partition λ of n. More information on this can be read in [4, §7–8].

6. Tensor Categories

For the following section, we will only be considering fields, $\mathbb{K}$, that are algebraically closed of characteristic zero, such as the field of complex numbers. We begin by first recalling the definition of a *category* and proceed to look at different types of categories. Specifically, we look at *monoidal categories* and work towards defining a particular type of monodial category, $\mathbb{K}$-*tensor category*, and then a *regular* $\mathbb{K}$-*tensor category*. More can be read on category theory throughout [5] and [3].

Definition 6.1 (Category)**.** A *category*, $\mathcal{C}$, consists of a class of objects, denoted $\mathrm{ob}(\mathcal{C})$, and a class of morphisms or maps between the objects, denoted $\mathrm{Hom}(\mathcal{C})$. We write $\mathrm{Hom}(a, b)$ or $\mathrm{Hom}_{\mathcal{C}}(a, b)$ for the class of all morphisms from a to b where $a, b \in \mathrm{ob}(\mathcal{C})$.

These collections must satisfy:

- For $a, b, c \in \mathrm{ob}(\mathcal{C})$, there is a composition map, $\mathrm{Hom}(a, b) \times \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$, defined by $g \circ f$ where $f \in \mathrm{Hom}(a, b)$, $g \in \mathrm{Hom}(b, c)$ and $g \circ f \in \mathrm{Hom}(a, c)$.
- For $f \in \mathrm{Hom}(a, b)$, $g \in \mathrm{Hom}(b, c)$, and $h \in \mathrm{Hom}(c, d)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.
- For any $a \in \mathrm{ob}(\mathcal{C})$, there is an *identity morphism*, id_a, such that for every morphism $f\colon x \to a$ and $g\colon a \to y$ we have $\mathrm{id}_a \circ f = f$ and $g \circ \mathrm{id}_a = g$.

Example 6.2 (Category of Super Vector Spaces)**.** Let $\mathbb{K}$-**SVect** be the category whose object are super vector spaces over some field $\mathbb{K}$ and morphisms are all even linear transformations between these SVS. We now check that composition is associative and there is identity map:

- The composition of morphisms will be the usual composition of linear transformations. Notice that if f, g are both even linear transformations then $f \circ g$ will also be even.
- Associativity is directly inherited from the associativity of composition of linear transformations.
- For any SVS $V \in \mathrm{ob}(\mathbb{K}\textbf{-SVect})$, the identity morphism is the identity transformation on V.

Definition 6.3 (Functor)**.** Let C and D be categories. A *functor* is a map $F\colon C \to D$ such that

- we have $F(X) \in \mathrm{ob}\, D$ for every $X \in \mathrm{ob}(C)$ and
- for all $f \in \mathrm{Hom}(C)$, where $f\colon X \to Y$, we have $F(f) \in \mathrm{Hom}(D)$ with $F(f)\colon F(X) \to F(Y)$ satisfying the following conditions:
  - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every $X \in \mathrm{ob}(C)$ and
  - $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathrm{Hom}(C)$.

Definition 6.4 (Equivalence of Categories)**.** Let C and D be categories. An *equivalence of categories* consists of two functors $F\colon C \to D$ and $G\colon D \to C$ and two natural isomorphisms $\varphi\colon FG \to \mathrm{id}_D$ and $\psi\colon \mathrm{id}_C \to GF$. If such functors and maps exist, then we say C and D are *equivalent*.

Definition 6.5 (Monoidal Category)**.** A *monoidal category* is a category $\mathcal{C}$ with a bifunctor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ called the tensor product or monoidal product such that there is

- a left and right identity, $I\in \mathrm{ob}(\mathcal{C})$, such that for all $A\in\mathcal{C}$, $I\otimes A\cong A$ and $A\otimes I\cong A$,
- for all $A,B,C\in \mathrm{ob}\,\mathcal{C}$ an isomorphism $\alpha_{A,B,C}\colon (A\otimes B)\otimes C\to A\otimes(B\otimes C)$, called an *associator*, and
- the following diagrams commute for all $A,B,C,D\in\mathcal{C}$:

$$\begin{tikzcd}
(A\otimes I)\otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr] & & A\otimes(I\otimes B) \arrow[dl] \\
& A\otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A\otimes(B\otimes C))\otimes D \arrow[dr, "\alpha_{A,(B\otimes C),D}"] & \\
((A\otimes B)\otimes C)\otimes D \arrow[ur, "\alpha_{A,B,C}\otimes\mathrm{id}_D"] \arrow[d, "\alpha_{(A\otimes B),C,D}"] & & A\otimes((B\otimes C)\otimes D) \arrow[d, "\mathrm{id}_A\otimes\alpha_{B,C,D}"] \\
(A\otimes B)\otimes(C\otimes D) \arrow[rr, "\alpha_{A,B,C\otimes D}"] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd}$$

Example 6.6. An example of a monoidal category is $\mathbb{K}$-SVect with standard tensor product as the monoidal product and the one dimensional even vector space $\mathbb{K}=\mathbb{K}_0$ being the identity, I, and lastly we need only define α. We define α for all $A,B,C\in\mathbb{K}$-SVect $\alpha_{A,B,C}\colon(A\otimes B)\otimes C\to A\otimes(B\otimes C)$ to be $\alpha_{A,B,C}(\sum(a_i\otimes b_i)\otimes c_i)=\sum a_i\otimes(b_i\otimes c_i)$ for all $\sum(a_i\otimes b_i)\otimes c_i\in(A\otimes B)\otimes C$. With these definitions, it is easy to check that the diagrams in Definition 6.5 commute.

Definition 6.7 (Braided Monodial Category)**.** We say a category, $\mathcal{C}$, is a *braided monoidal category* if for all $A,B\in\mathcal{C}$ there is an isomorphism, $\gamma_{A,B}\colon A\otimes B\to B\otimes A$, such that the following diagrams commute for all objects $A,B,C\in\mathcal{C}$:

$$\begin{tikzcd}
& (A\otimes B)\otimes C \arrow[dl, "\gamma_{A,B}\otimes\mathrm{id}_C"] \arrow[dr, "\alpha_{A,B,C}"] & \\
(B\otimes A)\otimes C \arrow[d, "\alpha_{B,A,C}"] & & A\otimes(B\otimes C) \arrow[d, "\gamma_{A,B\otimes C}"'] \\
B\otimes(A\otimes C) \arrow[dr, "\mathrm{id}_B\otimes\gamma_{A,C}"] & & (B\otimes C)\otimes A \arrow[dl, "\alpha_{B,C,A}"'] \\
& B\otimes(C\otimes A) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & A \otimes (B \otimes C) \arrow[dl, "\mathrm{id}_A \otimes \gamma_{B,C}"'] \arrow[dr, "\alpha^{-1}_{A,B,C}"] & \\
A \otimes (C \otimes B) \arrow[d, "\alpha^{-1}_{A,C,B}"] & & (A \otimes B) \otimes C \arrow[d, "\gamma_{A \otimes B, C}"'] \\
(A \otimes C) \otimes B \arrow[dr, "\gamma_{A,C} \otimes \mathrm{id}_B"] & & C \otimes (A \otimes B) \arrow[dl, "\alpha^{-1}_{C,A,B}"'] \\
 & (C \otimes A) \otimes B &
\end{tikzcd}$$

Definition 6.8 (Symmetric Monoidal Category)**.** A category, $\mathcal{C}$, is a *symmetric monoidal category* if it is a braided monoidal category that is maximally symmetric. That is, for all $A, B \in \mathcal{C}$ we have $\gamma_{A,B} \circ \gamma_{B,A} = \mathrm{id}_{A \otimes B}$, and for any $A, B, C \in \mathcal{C}$ the following diagrams commute:

$$\begin{tikzcd}
A \otimes I \arrow[rr, "\gamma_{A,I}"] \arrow[dr] & & I \otimes A \arrow[dl] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \otimes A \arrow[dr, "\gamma_{B,A}"] & \\
A \otimes B \arrow[ur, "\gamma_{A,B}"] \arrow[rr, equal, "\mathrm{id}_{A \otimes B}"'] & & A \otimes B
\end{tikzcd}$$

Example 6.9. A few simple examples of symmetric braided monodial categories are the following.

- The category of sets, denoted **Set**, is the category whose objects are sets and morphisms are functions between sets. This is an example of a monoidal category where $\otimes$ is just the Cartesian product and the unit object is any fixed singleton, in fact we can go farther and say that this is a symmetric braided monoidal category.
- The category of groups, denoted **Grp**, is the category whose objects are groups and morphisms are group homomorphisms. Similarly to **Set**, **Grp** is a symmetric braided monoidal category whose unit is the trivial group and $\otimes$ is standard Cartesian product.

Definition 6.10 (Rigid Monoidal Category)**.** An object A is *left rigid* if there is an object $B \in \mathcal{C}$ and morphisms $\eta_A \colon I \to A \otimes B$ and $\epsilon_A \colon B \otimes A \to I$ such that both of the following compositions are identities.

$$A \xrightarrow{\eta_A \otimes \mathrm{id}_A} (A \otimes B) \otimes A \xrightarrow{\alpha_{A,B,A}} A \otimes (B \otimes A) \xrightarrow{\mathrm{id}_A \otimes \epsilon_A} A$$

$$B \xrightarrow{\mathrm{id}_B \otimes \eta_A} B \otimes (A \otimes B) \xrightarrow{\alpha^{-1}_{B,A,B}} (B \otimes A) \otimes B \xrightarrow{\epsilon_A \otimes \mathrm{id}_B} B$$

We say B is *right rigid* if there exists A such that the above is satisfied. An object is said to be *rigid* if it is both left and right rigid. A *rigid monoidal category*, $\mathcal{C}$, is a monoidal category such that every object is rigid.

Example 6.11. Considering $\mathbb{K}$-SVect, we can show that $\mathbb{K}$-SVect is a rigid monidal category. Given any $V \in \mathbb{K}$-SVect we can pick a basis, B, and consider the dual space V^*. We let

$B^\vee = \{v^\vee \mid v \in B\}$ denote the dual basis of V^*. To show that V is left rigid, we define $\epsilon_V : V^* \otimes V \to \mathbb{K}$ for all $v \in V$, $f \in V^*$ via

$$\epsilon_V(f \otimes v) = f(v)$$

we extend this definition by linearity and define $\eta_V : \mathbb{K} \to V \otimes V^*$ for all $k \in \mathbb{K}$ via

$$\eta_V(k) = k \sum_{v \in B} v \otimes v^\vee.$$

Now it suffices to show that the compositions in Definition 6.10 are identities for all elements in B and $B^\vee$. We know that $\mathbb{K} \otimes V \cong V \cong V \otimes \mathbb{K}$. Now take any $k \in \mathbb{K}$ and $b \in B$ and notice that

$$\begin{aligned}
(\mathrm{id}_V \otimes \epsilon_V) \circ (\alpha_{V,V^*,V}) \circ (\eta_V \otimes \mathrm{id}_V)(k \otimes b) &= \mathrm{id}_V \otimes \epsilon_V(\alpha_{V,V^*,V}(\eta_V \otimes \mathrm{id}_V(k \otimes b))) \\
&= \mathrm{id}_V \otimes \epsilon_V(\alpha_{V,V^*,V}((k \sum_{v \in B} v_\otimes v^\vee) \otimes b)) \\
&= \mathrm{id}_V \otimes \epsilon_V(\sum_{v \in B} kv \otimes (v^\vee \otimes b)) \\
&= \sum_{v \in B} kv \otimes (v^\vee(b)) \\
&= kv \otimes 1 = v \otimes k \cong k \otimes v.
\end{aligned}$$

Notice that the second to last equality is given by that fact that $v^\vee(b) = 1$ only when $v = b$, otherwise $v^\vee(b) = 0$. Similarly consider for all $b^\vee \in B^\vee$ and $k \in \mathbb{K}$ then we have

$$\begin{aligned}
(\epsilon_V \otimes \mathrm{id}_{V^*}) \circ (\alpha^{-1}_{V^*,V,V^*}) \circ (\mathrm{id}_B \otimes \eta_V)(b^\vee \otimes k) &= \epsilon_V \otimes \mathrm{id}_{V^*}(\alpha^{-1}_{V^*,V,V^*}(\mathrm{id}_B \otimes \eta_V(b^\vee \otimes k))) \\
&= \epsilon_V \otimes \mathrm{id}_{V^*}(\alpha^{-1}_{V^*,V,V^*}(b^\vee \otimes (k \sum_{v \in B} v \otimes v^\vee))) \\
&= \epsilon_V \otimes \mathrm{id}_{V^*}(\sum_{v \in B} (b^\vee \otimes kv) \otimes v^\vee) \\
&= \sum_{v \in B} (k \cdot b^\vee(v)) \otimes v^\vee \\
&= k \otimes v^\vee \cong v^\vee \otimes k.
\end{aligned}$$

The final step is given again by $b^\vee(v) = 1$ only when $b = v$, otherwise $b^\vee(v) = 0$. Thus, given these definitions the diagrams, in Definition 6.10 are identities and therefore V is left rigid. We can use similar functions to show that V is also right rigid and therefore we can conclude that V is, in fact, rigid. This is for any $V \in \mathbb{K}$-SVect and therefore $\mathbb{K}$-SVect is a rigid monoidal category.

Definition 6.12 ($\mathbb{K}$-Linear Category)**.** Let $\mathbb{K}$ be some field. A category is said to be $\mathbb{K}$-*linear* if $\mathrm{Hom}_{\mathcal{C}}(a, b)$ is a $\mathbb{K}$-vector space for all $a, b \in \mathcal{C}$ and the composition map, $\circ\colon \mathrm{Hom}(b, c) \otimes \mathrm{Hom}(a, b) \to \mathrm{Hom}(a, c)$ defined for all $f, g \in \mathrm{Hom}_{\mathcal{C}}$ by $\circ(f, g) = f \circ g$, is bilinear.

Definition 6.13 ($\mathbb{K}$-Tensor Category)**.** A $\mathbb{K}$-*tensor category*, $\mathcal{C}$, is a $\mathbb{K}$-linear rigid symmetric braided monoidal category.

Example 6.14. We have already seen that $\mathbb{K}$-SVect is a monoidal category as seen in Example 6.6. To show that it is, in fact, a $\mathbb{K}$-tensor category, first recall Example 6.11 which showed us that $\mathbb{K}$-SVect is a rigid monoidal category. Now notice that for all $A, B \in \mathbb{K}$-SVect, we define $\gamma_{AB} \colon A \otimes B \to B \otimes A$ to be $\gamma_{AB}(a \otimes b) = b \otimes a$, extended by linearity. With this definition, the diagrams in Definition 6.5 and Defintion 6.8 will all commute, thus $\mathbb{K}$-SVect is a rigid symmetric braided monoidal category. Recall that $\mathscr{L}(V, W)$ is a $\mathbb{K}$-vector space and the standard composition of maps is bilinear. Therefore $\mathbb{K}$-SVect is a $\mathbb{K}$-linear rigid symmetric braided monoidal category. Thus, we conclude that $\mathbb{K}$-SVect is, in fact, a $\mathbb{K}$-tensor category.

Definition 6.15 (Schur Functor)**.** Let $\mathcal{C}$ be a $\mathbb{K}$-tensor category. Then for $X \in \mathcal{C}$, $n \in \mathbb{N}$ and a partition, λ, of n then we say the value of the *Schur Functor*, denoted S_λ, on X is

$$S_\lambda(X) := (V_\lambda \otimes X^{\otimes n})^{S_n} := \left(\frac{1}{n!} \sum_{g \in S_n} \rho(g) \right) (V_\lambda \otimes X^{\otimes n}),$$

where:

- the action ρ is the diagonal action arising from the standard action of S_n on V_λ and the action of S_n on $X^{\otimes n}$ described in Example 5.6 and
- $(-)^{S_n}$ denotes the subspace of invariants by action ρ.

Definition 6.16 (Subquotient)**.** We say an object is a *subobject* if it sits within another object of the same cateogry. A *subquotient* is a quotient object of a subobject.

Definition 6.17 (Finitely $\otimes$-Generated)**.** A category, $\mathcal{C}$, is *finitely $\otimes$-generated* if there exists $X \in \mathcal{C}$ such that every other object is isomorphic to a subquotient of a direct sum of objects $X^{\otimes n}$. If X is a such an object we call it a *$\otimes$-generator*.

Example 6.18. Some examples of finitely $\otimes$-generated categories are:

- The category $\mathbb{K}$-SVect of finite-dimensional SVS with the odd supervector space $\mathbb{K}$ as the $\otimes$-generator.
- The category $\operatorname{Rep}(G)$ of finite-dimensional representations of a finite group G.
- The category $\operatorname{Rep}(G)$ of finite-dimensional representations of an affine algebraic group G over $\mathbb{K}$. Any faithful representation X of G is the $\otimes$-generator of this category. We will not consider this example in further detail as it is beyond the scope of this paper.

Definition 6.19 (Regular $\mathbb{K}$-Tensor Category)**.** A $\mathbb{K}$-tensory category, $\mathcal{C}$, is called *normal* if

- it is finitely generated and
- for every object $Y \in \mathcal{C}$ there exits $n \in \mathbb{N}$ and a partition, λ, of n such that $S_\lambda(Y) = 0$.

7. H-Categories

In the following section, we take our previously defined super Hopf algebras, H, and look at and prove some properties held by the categories of finite-dimensional modules, H-**Mod**, and of finite-dimensional comodules, H-**Comod**. Lastly, we prove that the categories H-**Mod** and H^*-**Comod** are equivalent.

First, suppose $(H, \nabla, \eta, \Delta, \epsilon, S)$ is a Hopf algebra. We consider the category of finite-dimensional H-modules, denoted H-**Mod**. Now take M and N as finite-dimensional H-modules. Notice that $M \otimes N$ is also an H-module when we define for all $h \in H$, where $\Delta(h) = \sum h_1 \otimes h_2$, and for all $m \otimes n \in M \otimes N$

$$h(m \otimes n) = \Delta(h)(m \otimes n) = \left(\sum h_1 \otimes h_2\right)(m \otimes n) = \sum h_1 m \otimes h_2 n.$$

We continue by noticing that if M is an object of H-**Mod** then $M^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ is also an element of H-**Mod** with the following action. We define the action for all $m \in M$, $F \in M^*$ and $h \in H$ via

$$(hF)(m) = F(S(h)m).$$

Proposition 7.1. If $(H, \nabla, \eta, \Delta, \epsilon, S)$ is a Hopf algebra then the category of finite-dimensional H-modules is a $\mathbb{K}$-linear rigid monoidal category.

Proof. Notice that H-**Mod** forms a monoidal category with the definition above and $\mathbb{K}$ as the identity object. To show rigidity, take any $M \in H$-**Mod** with basis B. Now we consider M^* with the dual basis $B^\vee = \{m^\vee \mid m \in B\}$ and define $\epsilon_M \colon M^* \otimes M \to \mathbb{K}$ for all $f \otimes m \in M^* \otimes M$ via

$$\epsilon_M(f \otimes m) = f(m)$$

and extend by linearity. We define $\eta_M \colon \mathbb{K} \to M \otimes M^*$ for all $k \in \mathbb{K}$ via

$$\eta_M(k) = k \sum_{m \in B} m \otimes m^\vee.$$

With these definitions, one can prove that H-**Mod** is rigid monoidal category. The proof is done analously to Example 6.11 and is left to the reader. Lastly, the homomorphisms in H-**Mod** are module homomorphisms and therefore H-**Mod** is a $\mathbb{K}$-linear category. Thus we know that H-**Mod** is a $\mathbb{K}$-linear rigid monoidal category. ■

Proposition 7.2. If $(H, \nabla, \eta, \Delta, \epsilon, S)$ is a cocommutative Hopf algebra then H-**Mod** is a $\mathbb{K}$-tensor category.

Proof. Let $(H, \nabla, \eta, \Delta, \epsilon, S)$ be a cocommutative Hopf algebra. We know from Proposition 7.1 that H-**Mod** is a $\mathbb{K}$-linear rigid monoidal category. To show that H-**Mod** is a braided category, for any $M, N \in H$-**Mod** we define the map $\gamma_{M,N} \colon M \otimes N \to N \otimes M$ for all $m \otimes n \in M \otimes N$

$$\gamma_{M,N}(m \otimes n) = (n \otimes m) \text{ and extend by linearity.}$$

We need only show that $\gamma_{M,N}$ is an H-module homomorphism. To do so take any $m \otimes n \in M \otimes N$ and $h \in H$, where $\Delta(h) = \sum h_1 \otimes h_2$, then consider

$$\begin{aligned}
\gamma_{M,N}(h(m \otimes n)) &= \gamma_{M,N}\left(\Delta(h)(m \otimes n)\right) = \gamma_{M,N}\left(\left(\sum h_1 \otimes h_2\right)(m \otimes n)\right) \\
&= \gamma_{M,N}\left(\sum h_1 m \otimes h_2 n\right) = \sum h_2 n \otimes h_1 m \\
&= \left(\sum h_2 \otimes h_1\right)(n \otimes m) = \Delta(h)(\gamma_{M,N}(m \otimes n)) = h(\gamma_{M,N}(m \otimes n)).
\end{aligned}$$

Notice that the second-to-last equation is true since H is cocommutative. Therefore, we know that H-**Mod** is a braided monoidal category. Moreover, notice that $\gamma_{N,M} \circ \gamma_{M,N} = \mathrm{id}_{M,N}$.

Thus H-**Mod** is, in fact, a symmetric monoidal category. Finally, we can conclude that H-**Mod** is a $\mathbb{K}$-tensor category. ■

Definition 7.3 (Super Comodule)**.** Suppose (C, Δ, ϵ) is a super coalgebra over $\mathbb{K}$. A *right super comodule* over C is a SVS, M, together with $\rho\colon M \to M \otimes C$, called the *coaction* on M, such that ρ is even and the following diagrams commute.

$$\begin{tikzcd} M \otimes C \arrow[dr, "\mathrm{id}_M \otimes \Delta"'] & M \arrow[l, "\rho"'] \arrow[r, "\rho"] & M \otimes C \arrow[dl, "\rho \otimes \mathrm{id}_C"] \\ & M \otimes C \otimes C & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, equal, "\mathrm{id}_M"'] & M \otimes C \arrow[dl, "\mathrm{id}_M \otimes \epsilon"] \\ M & \end{tikzcd}$$

In other words, for all $m \in M$, $(\mathrm{id}_m \otimes \Delta) \circ \rho(m) = (\rho \otimes \mathrm{id}_C) \circ \rho(m)$ and $(\mathrm{id}_M \otimes \epsilon) \circ \rho(m) = \mathrm{id}_M(m)$, where we are identifying $M \otimes \mathbb{K} \cong M$. A *left comodule* is defined similarly to a right comodule. We denote the category of C-comodules by C-**Comod**.

Lemma 7.4. Let H and M both be finite-dimensional. The map $\omega\colon H \otimes M^* \to \mathrm{Hom}_{\mathbb{K}}(M, H)$ defined for all $\sum h_i \otimes f_i \in H \otimes M^*$ and $m \in M$ via $\omega(\sum h_i \otimes f_i)(m) = \sum f_i(m) h_i$ is an isomorphism.

Proof. This proof is left to reader. To sketch the proof, it is done by first noticing that $\dim(H \otimes M^*) = \dim(\mathrm{Hom}_{\mathbb{K}}(M, H))$ then it suffices to show that ω is injective or surjective. ■

It follows, we will view the map ω of Lemma 7.4 as an identification.

Proposition 7.5. Fix some finite-dimensional cocommutative super Hopf algebra, $(H, \nabla, \eta, \Delta, \epsilon, S)$. The category H-**Comod** is then a $\mathbb{K}$-tensor category.

Proof. Suppose $(H, \nabla, \eta, \Delta, \epsilon, S)$ is a finite-dimensional cocommutative super Hopf algebra. Notice that the morphisms of H-**Comod** are $\mathbb{K}$-vector spaces and therefore H-**Comod** is a $\mathbb{K}$-linear category.

We define the coaction on tensor products of comodules for all $M, N \in H$-**Comod** with coactions ρ_M and ρ_N. Now $M \otimes N$ is also an H-comodule with the coaction $\rho_{M \otimes N}$ defined as the composition of the following maps

$$M \otimes N \xrightarrow{\rho_M \otimes \rho_N} H \otimes M \otimes H \otimes N \xrightarrow{\mathrm{id}_H \otimes \mu \otimes \mathrm{id}_N} H \otimes H \otimes M \otimes N \xrightarrow{\nabla \otimes \mathrm{id}_{M \otimes N}} H \otimes M \otimes N,$$

where μ is the flip operator. With such a definition of the coaction on tensor products of comodules, H-**Comod** is a monoidal category. Now H-**Comod** is inherently a symmetric braided category. The proof of symmetry and braided left to the reader and is done similarly to Proposition 7.2.

To show that H-**Comod** is rigid, first let (M, ρ_M) be an object of H-**Comod**. Notice that (M^*, ρ_{M^*}) is an object of H-**Comod** when ρ_{M^*} is defined for all $f \in M^*$ and all $m \in M$ via

$$\rho_{M^*}(f)(m) = (S \otimes f) \circ \rho_M(m) = \sum S(h_i) \otimes f(m_i),$$

where $\rho_M(m) = \sum h_i \otimes m_i$. Now to show that any comodule, M, is left rigid, since M^* is an object of H-**Comod** we define $\epsilon_M\colon M \otimes M^* \to \mathbb{K}$ for all simple tensors $m \otimes f \in M \otimes M^*$ via $\epsilon_M(m \otimes f) = f(m)$ and extend by linearity. A similar map can be defined to show that M is right rigid. Thus we then know that H-**Comod** is also rigid and therefore a $\mathbb{K}$-tensor category. ■

Proposition 7.6. Fix some finite-dimensional super Hopf algebra $(H, \nabla, \eta, \Delta, \eta, S)$ then the categories H-**Mod** and H^*-**Comod** are equivalent.

Proof. Fix some finite-dimensional super Hopf algebra, $(H, \nabla, \eta, \Delta, \eta, S)$. We define the functors $F\colon H\text{-}\mathbf{Mod} \to H^*\text{-}\mathbf{Comod}$ and $G\colon H^*\text{-}\mathbf{Comod} \to H\text{-}\mathbf{Mod}$ via

$$F(M) = M^* \text{ and } G(N) = N^*,$$

for all $M \in H$-**Mod** and all $N \in H^*$-**Comod**. Similarly, for all $f \in \mathrm{Hom}(H\text{-}\mathbf{Mod})$ and all $g \in \mathrm{Hom}(H^*\text{-}\mathbf{Comod})$ we define

$$F(f) = f^* \text{ and } G(g) = g^*.$$

To check that both F and G are well defined, take $(M, \sigma) \in H$-**Mod** and consider (M^*, σ^*). Notice that taking the dual of M and the transpose of σ yields the following diagrams.

$$\begin{tikzcd} M^* \otimes H^* \arrow[dr, "\mathrm{id}_{M^*}\otimes\nabla^*"'] & M^* \arrow[l, "\sigma^*"'] \arrow[r, "\sigma^*"] & M^* \otimes H^* \arrow[dl, "\sigma^*\otimes\mathrm{id}_{H^*}"] \\ & M^* \otimes H^* \otimes H^* & \end{tikzcd} \qquad \begin{tikzcd} M^* \arrow[r, "\sigma^*"] \arrow[d, equal, "\mathrm{id}_{M^*}"'] & M^* \otimes H^* \arrow[dl, "\mathrm{id}^*_M\otimes\eta^*"] \\ M^* & \end{tikzcd}$$

Thus $F(M) \in H^*$-**Comod** and by the same process for $N \in H^*$-**Comod** then $G(N) \in H$-**Mod**. We know that taking the dual respects composition and the dual of the identity is the identity on the dual space. Thus F and G are, in fact, functors. Furthermore, we also know that for any vector space, M, and any linear map, f, then we naturally have $V \cong V^{**}$ and $f \cong f^{**}$. Thus H-**Mod** and H^*-**Comod** are equivalent. ∎

8. Supergroups

In our final section, we take all the concepts looked at throughout this paper and bring them together to define *supergroups* and finally *super-representations*. With these defined, we show that super-representations are $\mathbb{K}$-tensor categories. Lastly, we present Deligne's Theorem on Tensor Categories.

One source of Hopf algebras are the Hopf algebras $\mathcal{O}(G)$ of functions on a group G. Inspired by this, we use the associated terminology for an *arbitrary* super Hopf algebra.

Definition 8.1 (Affine Algebraic Supergroup)**.** An *affine algebraic supergroup* (*supergroup* for short), G, is the formal dual of a supercommutative super Hopf algebra. By this, we mean that we have a supercommutative super Hopf algebra $\mathcal{O}(G)$, and structures on G are defined to be the dual structures on $\mathcal{O}(G)$. For instance,

- a G-module is, by definition, an $\mathcal{O}(G)$-comodule.
- Similarly, the category $\mathrm{Rep}(G)$ of G-super modules is the category of super comodules over $\mathcal{O}(G)$.
- Any element of G_0 is an algebra homomorphism $\mathcal{O}(G) \to \mathbb{K}$.

Definition 8.2 (Parity Involution)**.** Given a super algebra its *parity involution* is the algebra automorphism which maps homogeneous elements, g, to $(-1)^{p(g)}g$.

Consider $\sigma \in G$. If G is a standard group then $\sigma^2 = 1$ makes sense but if G is a supergroup then there is more to define to make sense of $\sigma^2 = 1$. We say that the map σ satisfies $\sigma^2 = 1$ if for all $h \in \mathcal{O}(G)$

$$(\sigma \otimes \sigma) \circ (\Delta(h)) = \epsilon(h).$$

Now we define the *inner automorphism*, on a supergroup G, induced by σ as the map $\mathcal{O}(G) \to \mathcal{O}(G)$ defined via

$$(\sigma \otimes \mathrm{id}_{\mathcal{O}(G)} \otimes (\sigma \circ S)) \circ (\Delta \otimes \mathrm{id}_{\mathcal{O}(G)}) \circ (\Delta).$$

The inner automorphism from above corresponds to the parity involution if for all homogeneous $h \in \mathcal{O}(G)$

$$(\sigma \otimes \mathrm{id}_{\mathcal{O}(G)} \otimes (\sigma \circ S)) \circ (\Delta \otimes \mathrm{id}_{\mathcal{O}(G)}) \circ (\Delta(h)) = (-1)^{p(h)} h.$$

Definition 8.3 (Inner Parity)**.** An *inner parity* of a supergroup G is an element $\sigma \in G_0$ such that $\sigma^2 = 1$ and the inner automorphism of G induced by σ is the parity involution.

Let G be a supergroup and $\sigma \in G$ be an inner parity. For $(V, \rho) \in \mathrm{Rep}(G)$, we have the endomorphism, $\sigma\rho$, of V given by the composition

$$V \xrightarrow{\rho} \mathcal{O}(G) \otimes V \xrightarrow{\sigma \otimes \mathrm{id}_V} \mathbb{K} \otimes V \cong V.$$

Definition 8.4 (Super-Representation)**.** Given a supergroup G and an inner parity σ, then $\mathrm{Rep}(G, \sigma)$ is the category whose objects are objects, (V, ρ), of $\mathrm{Rep}(G)$ such that $\sigma\rho$ is the parity endomorphism (i.e. for all homogeneous $v \in V$, $(\sigma \otimes \mathrm{id}_V) \circ \rho(v) = (-1)^{p(v)} v$).

Proposition 8.5. Let G be a supergroup and σ be an inner parity. The category $\mathrm{Rep}(G, \sigma)$ is a $\mathbb{K}$-tensor category.

Proof. Let G be a super group and let $\sigma \colon \mathcal{O}(G) \to \mathbb{K}$ be an inner parity. Consider the category $\mathrm{Rep}(G, \sigma)$. By Proposition 7.2 the category $\mathrm{Rep}(G)$ is a $\mathbb{K}$-tensor category. If we can show that $\mathrm{Rep}(G, \sigma)$ closed under tensor product and that it is also rigid then it is also a $\mathbb{K}$-tensor category. Take any two $M, N \in \mathrm{Rep}(G, \sigma)$. We know that the action of σ on both M and N is the parity endomorphism. Take any $m \in M$ and $n \in N$. We know that

$$(\sigma \otimes \mathrm{id}_M) \circ \rho_M(m) = (\sigma \otimes \mathrm{id}_M) \left(\sum h'_i \otimes m_i \right) = (-1)^{p(m)} m$$

and

$$(\sigma \otimes \mathrm{id}_N) \circ \rho_N(n) = (\sigma \otimes \mathrm{id}_N) \left(\sum h''_j \otimes n_j \right) = (-1)^{p(n)} n.$$

Now we consider the H-comodule $M \otimes N$. Since σ is an algebra homomorphism, we have that $\sigma(\nabla(h_1 \otimes h_2)) = \sigma(h_1)(h_2)$. Thus we have for simple tensors

$$\begin{aligned}
(\sigma \otimes \mathrm{id}_{M\otimes N}) \circ \rho_{M\otimes N}(m \otimes n) &= (\sigma \otimes \mathrm{id}_{M\otimes N}) \circ (\nabla \otimes \mathrm{id}_{M\otimes N}) \circ (\mathrm{id}_H \otimes \mu \otimes \mathrm{id}_N) \circ (\rho_M(m) \otimes \rho_N(n)) \\
&= (\sigma \otimes \mathrm{id}_{M\otimes N}) \circ (\nabla \otimes \mathrm{id}_{M\otimes N}) \circ \left(\sum h'_i \otimes h''_j \otimes m_i \otimes n_j \right) \\
&= (\sigma \otimes \mathrm{id}_{M\otimes N}) \circ \left(\sum h'_i h''_j \otimes m_i \otimes n_j \right) \\
&= \sum \sigma(h'_i)\sigma(h''_j) \otimes m_i \otimes n_j \\
&= (-1)^{p(m)} m \otimes (-1)^{p(n)} n \\
&= (-1)^{p(m\otimes n)} m \otimes n.
\end{aligned}$$

We now extend this by linearity and we then know that $M \otimes N$ is in $\mathrm{Rep}(G, \sigma)$. Thus $\mathrm{Rep}(G, \sigma)$ is a $\mathbb{K}$-linear symmetric braided monoidal category.

To show that $\mathrm{Rep}(G, \sigma)$ is rigid, we need only show that if $M \in \mathrm{Rep}(G, \sigma)$ then $M^* \in \mathrm{Rep}(G, \sigma)$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & M \arrow[rr, "\rho"] & & \mathcal{O}(G) \otimes M \arrow[dr, "\sigma \otimes \mathrm{id}_M"] & \\
M \arrow[r, "\rho"] \arrow[rrrr, bend right=30, "\mathrm{id}_M"'] & \mathcal{O}(G) \otimes M \arrow[u, "\sigma S \otimes \mathrm{id}_M"] \arrow[rr, bend left, "\mathrm{id}_{\mathcal{O}(G)} \otimes \rho"] \arrow[rr, bend right, "\Delta \otimes \mathrm{id}_M"'] \arrow[rrr, bend right=25, "\epsilon \otimes \mathrm{id}_M"'] & & \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes M \arrow[u, "\mathrm{id}_{\mathcal{O}(G)} \otimes \sigma S \otimes \mathrm{id}_M"'] \arrow[r, "\sigma \otimes \sigma S \otimes \mathrm{id}_M"'] & M
\end{tikzcd}$$

Notice that the action of σ on M (i.e. $(\sigma \otimes \mathrm{id}_M) \circ \rho$) is the inverse of the action of σS on M (i.e. $(\sigma S \otimes \mathrm{id}_M) \circ \rho$). Since the parity isomorphism is its own inverse and σ acts by the parity isomorphisms then σS acts also by the parity isomorphism. Knowing this, we now consider the dual comodule (M^*, ρ^*) where $\rho^* \colon M^* \to \mathcal{O}(G) \otimes M^* \cong \mathrm{Hom}_{\mathbb{K}}(M, \mathcal{O}(G))$ is defined as $\rho^*(f) = (f \otimes S) \circ \rho$ for all $f \in M^*$. Finally we compute the action of σ on ρ^*.

$$f \xrightarrow{\rho^*} (S \otimes f) \circ \rho \xrightarrow{\sigma \otimes \mathrm{id}_{M^*}} \sigma \circ (S \otimes f) \circ \rho = (\sigma S \otimes f) \circ \rho = f \circ (\sigma S \otimes \mathrm{id}_M) \circ \rho$$

We already know that $(\sigma S \otimes \mathrm{id}_M) \circ \rho$ acts as the parity isomorphism thus it is easy to check that this action is the parity isomorphism of f. Thus we have shown that if $M \in \mathrm{Rep}(G, \sigma)$ then so is M^*. Therefore, we know that $\mathrm{Rep}(G, \sigma)$ is rigid. Furthermore, we can conclude that $\mathrm{Rep}(G, \sigma)$ is then a $\mathbb{K}$-tensor category. ■

The following theorem is the primary motivation for this paper. It is referred to as Deligne's Theorem on Tensor Categories and was first seen and proven by Deligne in [2].

Theorem 8.6. Every regular $\mathbb{K}$-tensor category is equivalent to one of the form $\mathrm{Rep}(G, \sigma)$, for some supergroup G and some inner parity σ.

Deligne's Theorem on Tensor Categories highlights the importance of supergroups in both math and physics. In particular, regular tensor categories are ubiquitous in these subjects, and Deligne's Theorem tells us that we lose no generality in assuming that such categories are categories of representations of super groups.

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