LECTURES ON HEISENBERG AND KAC–MOODY CATEGORIFICATION

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Abstract. These are notes for a lecture series given at Beijing Institute of Technology, June 17–22, 2024. We explain how module categories over the Heisenberg category can be viewed as 2-representations over a Kac–Moody 2-category.

Contents

1. Introduction

The goal of these notes is to describe a precise relationship between representations of Heisenberg categories and Kac–Moody 2-categories. This relationship is described in full detail in [BSW20a]. We will start with a brief introduction to string diagrams in monoidal categories, which is the language that will be used throughout the notes. After this background, we introduce the Heisenberg category and the Kac–Moody 2-category, before stating the main result that relates the two. Our aim is not to present the results of [BSW20a] in full generality, nor is it to present the full details of all proofs. Instead, we will make several simplifying assumptions, in order to make the exposition as accessible as possible, and we will often give proofs of special cases that are computationally easier but give a good idea of the general argument. We will also try to demonstrate how a generating function approach to the diagrammatic categories involved allows one to make efficient computations in these categories.

Throughout these notes, to simplify the exposition, $\Bbbk$ denotes an algebraically closed field of characteristic zero. All vector spaces, algebras, categories, and functors will be assumed to be linear over $\Bbbk$ unless otherwise specified. (The results can be easily generalized to an arbitrary algebraically closed field.) We let $\mathbb{N} = \mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers.

2. Strict monoidal categories and string diagrams

In this section, we briefly review the definition of strict linear monoidal categories. These will be our main tool for exploring the concepts of symmetry and duality. We then discuss string diagrams, which are, in many ways, the best language to use when working with monoidal categories.

2.1. **Definitions.** Throughout this document, all categories are assumed to be locally small. In other words, we have a *set* of morphisms between any two objects.

A *strict monoidal category* is a category $\mathcal{C}$ equipped with

- a bifunctor (the *tensor product*) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and
- a *unit object* $\mathbb{1}$

such that, for all objects X, Y, and Z of $\mathcal{C}$, we have

- $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and
- $\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$,

and, for all morphisms f, g, and h of $\mathcal{C}$, we have

- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ and
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Here, and throughout the document, 1_X denotes the identity endomorphism of an object X.

Remark 2.1. Note that, in a (not necessarily strict) *monoidal category*, the equalities above are replaced by isomorphism, and one imposes certain coherence conditions. For example, let $\mathcal{V}ec_{\Bbbk}$ be the category of finite-dimensional $\Bbbk$-vector spaces. In this category one has isomorphisms $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$, but these isomorphisms are not equalities in general. Similarly, the unit object in this category is the one-dimensional vector space $\Bbbk$, and we have $\Bbbk \otimes V \cong V \cong V \otimes \Bbbk$ for any vector space V.

We will be building monoidal categories "from scratch" via generators and relations. Thus, we are free to require them to be strict. In general, Mac Lane's coherence theorem for monoidal categories asserts that every monoidal category is monoidally equivalent to a strict one. (For a proof of this fact, see [Mac98, §VII.2] or [Kas95, §XI.5].) So, in practice, we do not lose much by assuming monoidal categories are strict. (See also [Sch01].)

A $\Bbbk$-*linear category* is a category $\mathcal{C}$ such that

- for any two objects X and Y of $\mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a $\Bbbk$-module,
- composition of morphisms is bilinear:
$$f \circ (\alpha g + \beta h) = \alpha(f \circ g) + \beta(f \circ h),$$
$$(\alpha f + \beta g) \circ h = \alpha(f \circ h) + \beta(g \circ h),$$
for all $\alpha, \beta \in \Bbbk$ and morphisms f, g, and h such that the above operations are defined.

The category $\mathcal{V}ec_{\Bbbk}$ is an example of a $\Bbbk$-linear category. For any two $\Bbbk$-modules M and N, the space $\mathrm{Hom}_{\Bbbk}(M, N)$ is again a $\Bbbk$-module under the usual pointwise operations. Composition is bilinear with respect to this $\Bbbk$-module structure.

A *strict* $\Bbbk$*-linear monoidal category* is a category that is both strict monoidal and $\Bbbk$-linear, and such that the tensor product of morphisms is $\Bbbk$-bilinear.

One important property of monoidal categories is the *interchange law*. Suppose

$$X_1 \xrightarrow{f} X_2 \qquad \text{and} \qquad Y_1 \xrightarrow{g} Y_2$$

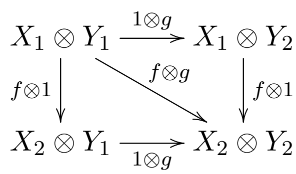
are morphisms in a strict $\Bbbk$-linear monoidal category $\mathcal{C}$. Then

$$(1_{X_2} \otimes g) \circ (f \otimes 1_{Y_1}) = \otimes((1_{X_2}, g)) \circ \otimes((f, 1_{Y_1})) = \otimes((1_{X_2}, g) \circ (f, 1_{Y_1})) = \otimes((f, g)) = f \otimes g,$$

where the second equality uses that the tensor product is a bifunctor. Similarly,

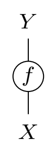
$$(f \otimes 1_{Y_2}) \circ (1_{X_1} \otimes g) = f \otimes g.$$

Thus, the following diagram commutes:

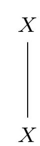


2.2. **String diagrams.** Strict monoidal categories are especially well suited to being depicted using the language of *string diagrams*. These diagrams, which are sometimes also called *Penrose diagrams*, have their origins in work of Roger Penrose in physics [Pen71]. Working with strings diagrams helps build intuition. It also often makes certain arguments obvious, whereas the corresponding algebraic proof can be a bit opaque. We give here a brief overview of string diagrams, referring the reader to [TV17, Ch. 2] for a detailed treatment. Throughout this section, $\mathcal{C}$ will denote a strict $\Bbbk$-linear monoidal category.

We will denote a morphism $f\colon X \to Y$ by a strand with a coupon labeled f:

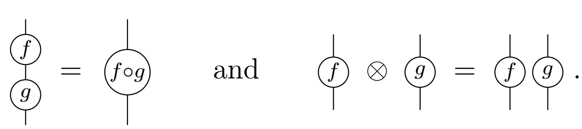


Note that we are adopting the convention that diagrams should be read from bottom to top, i.e., the domain is at the bottom and the codomain is at the top. The *identity map* $1_X\colon X \to X$ is a string with no coupon:

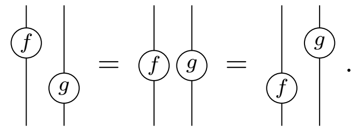


We sometimes omit the object labels (e.g. X and Y above) when they are clear or unimportant. We will also sometimes distinguish identity maps of different objects by some sort of decoration of the string (orientation, dashed versus solid, etc.), rather than by adding object labels.

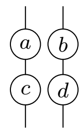
Composition is denoted by *vertical stacking* (recall that we read pictures bottom to top) and tensor product is *horizontal juxtaposition*:



The *interchange law* then becomes

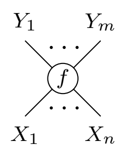


This graphical interpretation of the interchange law is one of the main reasons that the two-dimensional notation of string diagrams works so well for monoidal categories. Much as we may omit parenthesis when multiplying several elements in an associative algebra, string diagrams allow us to draw a single diagram



without specifying if this denotes $(a \otimes b) \circ (c \otimes d)$ or $(a \circ c) \otimes (b \circ d)$, since both expressions are equal.

A general morphism $f \colon X_1 \otimes \cdots \otimes X_n \to Y_1 \otimes \cdots \otimes Y_m$ can be depicted as a coupon with n strands emanating from the bottom and m strands emanating from the top:



3. Duality in monoidal categories

We now turn our attention to the concept of duality in monoidal categories. This is a general categorical concept that includes duals of modules and adjoint functors as examples.

3.1. **Duals in monoidal categories.** Suppose a strict monoidal category has two objects $\uparrow$ and $\downarrow$. Recalling our convention that we do not draw the identity morphism of the unit object $\mathbb{1}$, a morphism $\downarrow \otimes \uparrow \to \mathbb{1}$ would have string diagram

$$\curvearrowleft \colon \downarrow \otimes \uparrow \to \mathbb{1},$$

where we may decorate the cap with some symbol if we have more than one such morphism. The fact that the top of the diagram is empty space indicates that the codomain of this morphism is the unit object $\mathbb{1}$. Similarly, we can have

$$\circlearrowleft \colon \mathbb{1} \to \uparrow \otimes \downarrow.$$

We say that $\downarrow$ is *left dual* to $\uparrow$ (and $\uparrow$ is *right dual* to $\downarrow$) if we have morphisms

$$\curvearrowleft \colon \downarrow \otimes \uparrow \to \mathbb{1} \qquad \text{and} \qquad \circlearrowleft \colon \mathbb{1} \to \uparrow \otimes \downarrow$$

such that

(3.1)


The morphisms $\circlearrowleft$ and $\curvearrowleft$ are called the *unit* and the *counit*, respectively, of the duality. (The relations (3.1) are a generalization of the unit-counit formulation of adjunction of functors.) A monoidal category in which every object has both left and right duals is called a *rigid*, or *autonomous*, category.

If $\uparrow$ and $\downarrow$ are both left and right dual to each other, then, in addition to the above, we also have

$$\curvearrowright \colon \uparrow \otimes \downarrow \to \mathbb{1} \qquad \text{and} \qquad \circlearrowright \colon \mathbb{1} \to \downarrow \to \uparrow$$

such that

(3.2) 

To give a concrete example of duality in a monoidal category, consider the category $\mathcal{V}ec_{\Bbbk}$ of finite-dimensional $\Bbbk$-vector spaces. (The category $\mathcal{V}ec_{\Bbbk}$ is not strict, but this will not cause any problems for us. See Remark 2.1.) In this category, the unit object is $\Bbbk$. We claim that, if V is any finite-dimensional $\Bbbk$-vector space, the dual vector space V^* is left and dual to V in the sense defined above.

Define the *evaluation map*

$$\mathrm{ev}_V \colon V^* \otimes V \to \Bbbk, \qquad f \otimes v \mapsto f(v), \tag{3.3}$$

and the *coevaluation map*

$$\mathrm{coev}_V \colon \Bbbk \to V \otimes V^*, \qquad 1 \mapsto \sum_{v \in \mathbf{B}_V} v \otimes \delta_v, \tag{3.4}$$

where $\mathbf{B}_V$ is a basis of V and $\{\delta_v : v \in \mathbf{B}_V\}$ is the dual basis of V^*.

Taking $\uparrow = V$ and $\downarrow = V^*$, we define

$$\curvearrowleft = \mathrm{ev}_V \qquad \text{and} \qquad \circlearrowright = \mathrm{coev}_V\,.$$

Let us check the left-hand relation in (3.1). The left-hand side is the composition

$$V \cong \Bbbk \otimes V \xrightarrow{\circlearrowright \otimes 1_V} V \otimes V^* \otimes V \xrightarrow{1_V \otimes \curvearrowleft} V \otimes \Bbbk \cong V,$$
$$w \mapsto 1 \otimes w \mapsto \sum_{v \in \mathbf{B}_V} v \otimes \delta_v \otimes w \mapsto \sum_{v \in \mathbf{B}_V} \delta_v(w) \otimes v \mapsto \sum_{v \in \mathbf{B}_V} \delta_v(w) v = w.$$

Thus, this composition is precisely the identity map 1_V, and so the right-hand relation in (3.1) is satisfied. The verification of the left-hand equality in (3.1) is analogous and is left as an exercise for the reader.

In fact, V^* is also *right* dual to V. This can be shown directly by computations analogous to those above, or can be seen as a consequence of the more general result in symmetric monoidal categories.

Exercise 3.1. Show that the coevaluation map coev_V defined in (3.4) is independent of the choice of basis $\mathbf{B}_V$.

From now on, cups and caps in string diagrams will always denote units and counits giving the data of duality between objects.

Exercise 3.2. Units and counits are not unique. For instance, if we fix $\alpha \in \Bbbk^\times$, then $\alpha\, \mathrm{coev}_V$ and $\alpha^{-1}\, \mathrm{ev}_V$ are also units and counits expressing that V^* is left dual to a finite-dimensional vector space V. However, fixing the unit uniquely determines the counit, and vice versa. Indeed, let $\uparrow$ and $\downarrow$ be objects in a monoidal category $\mathcal{C}$.

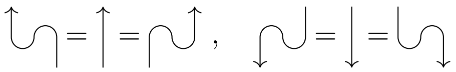
(a) Suppose that $\curvearrowleft \colon \downarrow \otimes \uparrow \to \mathbb{1}$ is a morphism in $\mathcal{C}$. Show that there exists at most one morphism $\circlearrowright \colon \mathbb{1} \to \uparrow \otimes \downarrow$ satisfying (3.1).
(b) Suppose that $\circlearrowright \colon \mathbb{1} \to \uparrow \otimes \downarrow$ is a morphism in $\mathcal{C}$. Show that there exists at most one morphism $\curvearrowleft \colon \downarrow \otimes \uparrow \to \mathbb{1}$ satisfying (3.1).

3.2. **A universal category with duals.** The *oriented Temperley–Lieb category* $\mathcal{OTL}_\Bbbk$ is the strict $\Bbbk$-linear monoidal category with

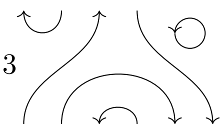
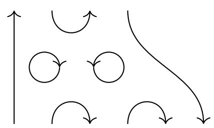
- two generating objects $\uparrow$ and $\downarrow$;
- four generating morphisms,

 $: \downarrow \otimes \uparrow \to \mathbb{1},$  $: \mathbb{1} \to \uparrow \otimes \downarrow,$  $: \uparrow \otimes \downarrow \to \mathbb{1},$  $: \mathbb{1} \otimes \downarrow \otimes \uparrow;$

- and relations



Objects in $\mathcal{OTL}_\Bbbk$ are finite tensor products of $\uparrow$ and $\downarrow$. An example of a morphism in $\mathcal{OTL}_\Bbbk$ is

3  −  $\in \mathrm{Hom}_{\mathcal{OTL}_\Bbbk}(\uparrow \otimes \uparrow \otimes \downarrow \otimes \uparrow \otimes \downarrow \otimes \downarrow, \uparrow \otimes \downarrow \otimes \uparrow \otimes \downarrow).$

The category $\mathcal{OTL}_\Bbbk$ is the free $\Bbbk$-linear monoidal category on one object with a two-sided dual. If X and Y are two objects in a $\Bbbk$-linear monoidal category $\mathcal{C}$ that are both left and right dual to each other, then there exists a unique monoidal functor

$$\mathcal{OTL}_\Bbbk \to \mathcal{C}, \qquad \uparrow \mapsto X, \qquad \downarrow \mapsto Y,$$

sending , , , and  to the units and counits of the dualities between X and Y.

Exercise 3.3. Show that $\mathcal{OTL}_\Bbbk$ is a rigid monoidal category. *Hint*: To show that arbitrary objects have duals, nest cups and caps.

4. The Heisenberg category

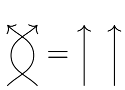
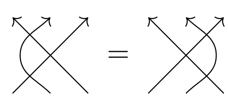
In this section, we introduce one of our main categories of interest: the Heisenberg category. The name comes from the fact that its Grothendieck ring is isomorphic to the infinite-rank Heisenberg algebra, although we will not discuss that property here. A special case of the Heisenberg category was first introduced by Khovanov in [Kho14]. The definition was then generalized in [MS18, Bru18].

4.1. **The symmetric group algebra category.** Define $\mathcal{S}ym$ to be the strict $\Bbbk$-linear monoidal category with:

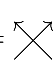
- one generating object $\uparrow$;
- one generating morphism

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow;$

- and relations

(4.1)  and .

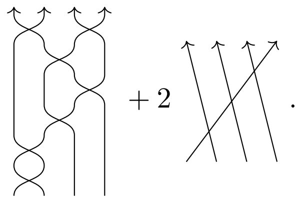
One could write these relations in a more traditional algebraic manner, if so desired. For example, if we let

$s =$  $: \uparrow \otimes \uparrow \ \to \ \uparrow \otimes \uparrow,$

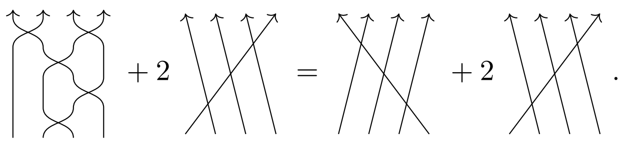
then the two relations (4.1) become

$$s^2 = 1_{\uparrow \otimes \uparrow} \qquad \text{and} \qquad (s \otimes 1_\uparrow) \circ (1_\uparrow \otimes s) \circ (s \otimes 1_\uparrow) = (1_\uparrow \otimes s) \circ (s \otimes 1_\uparrow) \circ (1_\uparrow \otimes s).$$

The objects of $\mathcal{S}ym$ are $\uparrow^{\otimes n}$, $n \in \mathbb{N}$. An example of an endomorphism of $\uparrow^{\otimes 4}$ is



Using the relations, we see that this morphism is equal to



Fix a positive integer n and recall that the group algebra $\Bbbk\mathfrak{S}_n$ of the symmetric group on n letters has a presentation with generators $s_1, s_2, \dots, s_{n-1}$ (the simple transpositions) and relations

(4.2) $$s_i^2 = 1, \qquad 1 \le i \le n-1,$$

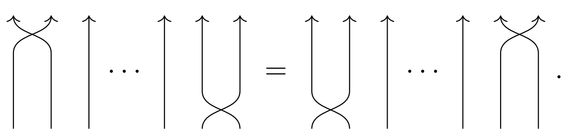
(4.3) $$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad 1 \le i \le n-2,$$

(4.4) $$s_i s_j = s_j s_i, \qquad 1 \le i, j \le n-1,\ |i-j| > 1.$$

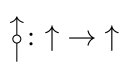
We have an isomorphism of algebras

(4.5) $$\Bbbk\mathfrak{S}_n \to \mathrm{End}_{\mathcal{S}ym}(\uparrow^{\otimes n})$$

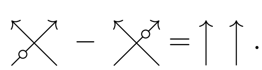
where s_i is sent to the crossing of the i-th and $(i+1)$-st strands, labeled from right to left. Note that the "distant braid relation" (4.4) follows for free from the interchange law:



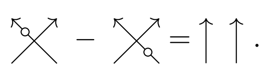
4.2. **The degenerate affine Hecke algebra category.** Define $\mathcal{DAH}$ to be the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{S}ym$ by adjoining one additional generating morphism

(4.6) 

and imposing the additional relation

(4.7) 

Exercise 4.1. Show that

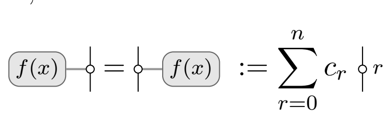
(4.8) 

The endomorphism algebra

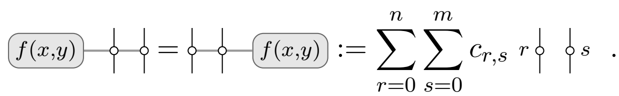
$$\mathrm{End}_{\mathcal{DAH}}(\uparrow^{\otimes n})$$

is isomorphic to the *degenerate affine Hecke algebra* of rank n.

When a dot is labelled by a multiplicity, we mean to take its power under vertical composition. For a polynomial $f(x) = \sum_{r=0}^{n} c_r x^r$, we use the shorthand

(4.9) 

to "pin" $f(x)$ to a dot on a string. Similarly, for $f(x,y) = \sum_{r=0}^{n} \sum_{s=0}^{m} c_{r,s} x^r y^s$, we use

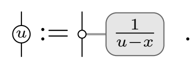
(4.10) 

Thus, the first variable, x, corresponds to the left dot and the second variable, y, corresponds to the right one.

We will often work with generating functions in an indeterminate u. We view

$$\frac{1}{u-x} = \sum_{n \geq 0} x^n u^{-n-1} = u^{-1} + u^{-2}x + u^{-3}x^2 + \cdots \in \Bbbk[x][\![u^{-1}]\!]$$

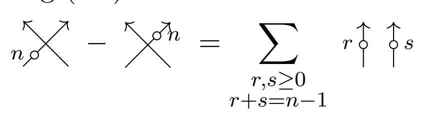
as a generating function for multiple dots on a string. Since this power series will appear frequently, we introduce the notation

(4.11) 

Lemma 4.2. *We have*

(4.12) 

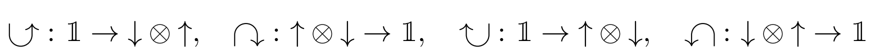
Proof. An inductive argument using (4.7) shows that

(4.13) 

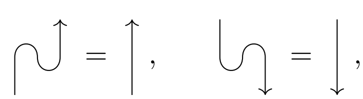
Multiplying by u^{-n-1} and summing over $n \in \mathbb{N}$ then gives the first equation in (4.12). The second equation in (4.12) is proved similarly, using (4.8). □

We see above the efficiency of the generating function approach. The simple relation (4.12) encodes the infinitely many relations (4.13).

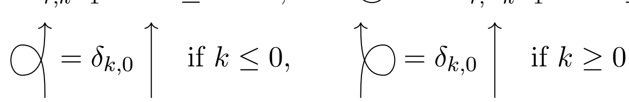
4.3. **The Heisenberg category.** The *Heisenberg category* $\mathcal{H}eis_k$ of central charge $k \in \mathbb{Z}$ is the strict monoidal category obtained from $\mathcal{DAH}$ by adjoining a generating object $\downarrow$ and morphisms

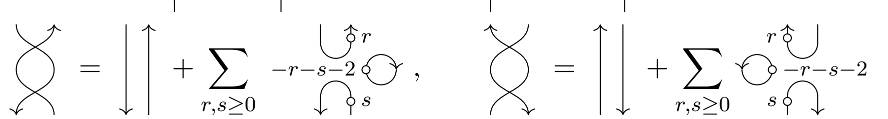


subject to the relations

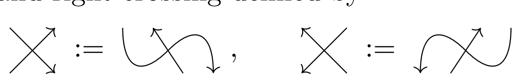
(4.14) 

(4.15) 

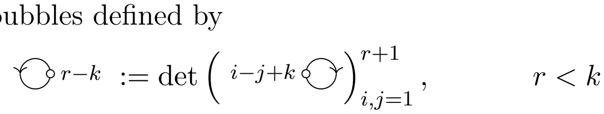
(4.16) 

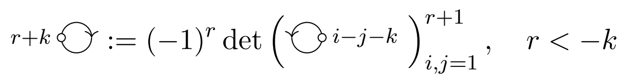
(4.17) 

where we are using the left and right crossing defined by

(4.18) 

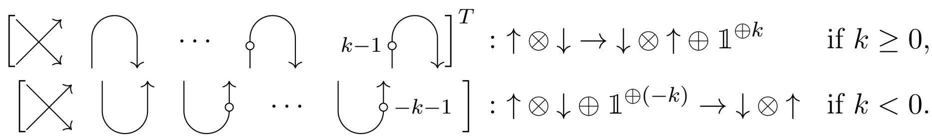
and negatively dotted bubbles defined by

(4.19) 

(4.20) 

where we interpert the determinants as 1 when $r = -1$ and as 0 when $r < -1$.

Remark 4.3. Relations (4.15) to (4.17) are equivalent to the statement that the following (which are matrices) is an isomorphism in the additive envelope of $\mathcal{H}eis_k$:

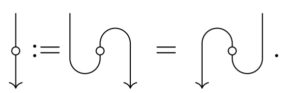


This implies that, in the Grothendieck ring of $\mathcal{H}eis_k$, we have

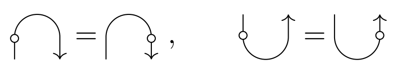
$$[\uparrow][\downarrow] = [\downarrow][\uparrow] + k,$$

which corresponds to the *canonical commutation relation* in the Heisenberg algebra.

One can show that $\uparrow$ is both left and right dual to $\downarrow$, and that $\mathcal{H}eis_k$ is *pivotal*, which means that isotopic diagrams represent the same morphism. We define dots on downward strands by



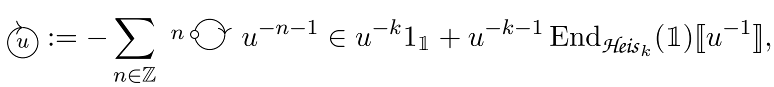
It follows that dots slide over caps and cups:

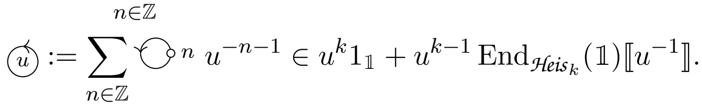


Thus, we also have

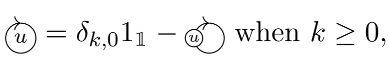
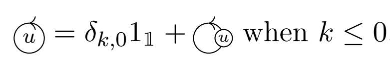
Much like (4.12), we can use generating functions to succinctly state important relations that hold in $\mathcal{H}eis_k$. For a Laurent series $p(u)$, we let $[p(u)]_{u^r}$ denote its u^r-coefficient, and we write $[p(u)]_{u<0}$ for $\sum_{n<0}[p(u)]_{u^n}u^n$.

When working with dotted bubbles, it is often helpful to assemble them all into a single generating function. Let

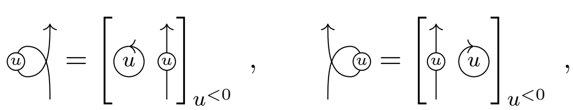
(4.21) 

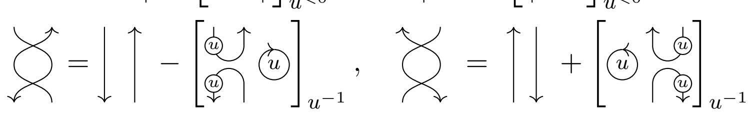
(4.22) 

Note that

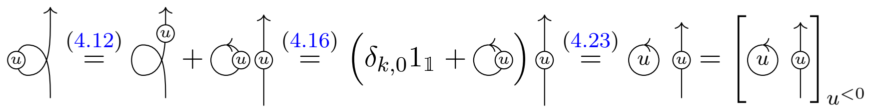
(4.23)  and 

Proposition 4.4. *The following relations hold in $\mathcal{H}eis_k$:*

(4.24) 

(4.25) 

Proof. The relations (4.25) are generating-function forms of (4.17). We give the proof of the first equation in (4.24) when $k \leq 0$. In this case, we have



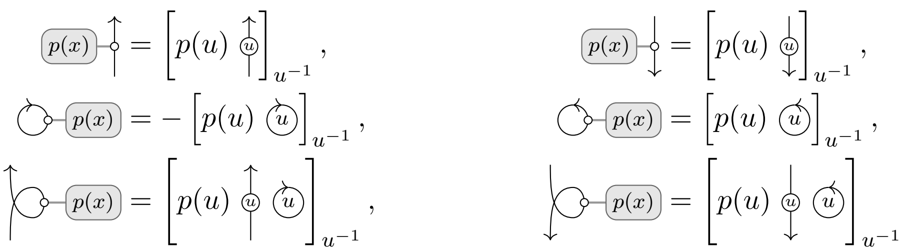
where the last equality follows from the fact that all powers of u appearing in the penultimate expression are negative. ☐

Remark 4.5. It follows from (4.21), (4.22), and (4.25) that

(4.26)

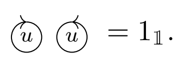

When $k = 0$, the category $\mathcal{H}eis_0$ is the *affine oriented Brauer category.*

Lemma 4.6. *For a polynomial $p(u) \in \Bbbk[u]$, we have*

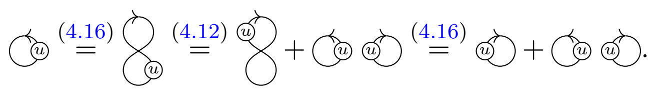
(4.27)
(4.28)
(4.29)


Proof. By linearity, it suffices to prove (4.27) and (4.28) in the case that $p(u) = u^r$ for $r \geq 0$. In that case, they follow easily after computing the u^{-1}-coefficient on the right-hand side, using (4.21) and (4.22). To prove (4.29), rewrite the left-hand side using (4.27), then apply the curl relation (4.24) ☐

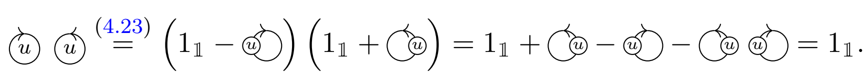
Proposition 4.7 (Infinite grassmannian relation)**.** *The following relation holds:*

(4.30)


Proof. We prove the special case $k = 0$. The general case is similar but slightly more complicated. We compute



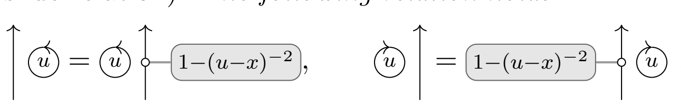
Therefore,



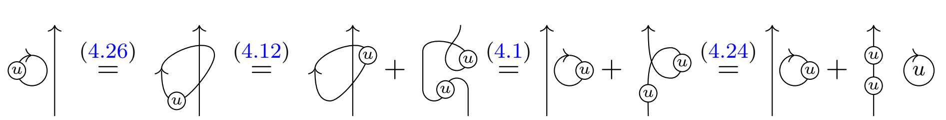
☐

The infinite grassmannian relation (4.30) allows one to write clockwise bubbles in terms of counterclockwise ones and vice versa. The following result will be crucial for us.

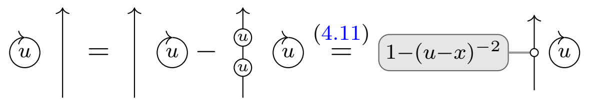
Lemma 4.8 (Bubble slide relation)**.** *The following relation holds:*

(4.31)


Proof. We prove the result in the case $k \geq 0$. The general case is similar but slightly more complicated.



Multiplying both sides by -1, adding $1_{\mathbb{1}}$ to both sides, and using (4.23) gives



This proves the second relation in (4.31). Multiplying both sides of this relation on the left and the right by (u) and then using the infinite grassmannian relation (4.30) gives the first relation in (4.31). ☐

Note that

$$1-(u-x)^{-2}=\frac{(u-(x+1))(u-(x-1))}{(u-x)^2}. \tag{4.32}$$

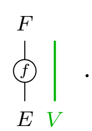
This will play an important role in the sequel.

5. Heisenberg module categories

A *module category* over $\mathcal{H}eis_k$ is a $\Bbbk$-linear category $\mathcal{R}$ together with a strict $\Bbbk$-linear monoidal functor

$$\mathsf{R}\colon \mathcal{H}eis_k \to \mathcal{E}nd_{\Bbbk}(\mathcal{R}),$$

where $\mathcal{E}nd_{\Bbbk}(\mathcal{R})$ denotes the strict $\Bbbk$-linear with objects that are $\Bbbk$-linear endofunctors of $\mathcal{R}$ and morphisms that are natural transformations. We usually suppress the monoidal functor R, using the same notation $f\colon E\to F$ both for a morphism in $\mathcal{H}eis_k$ and for the natural transformation between endofunctors of $\mathcal{R}$ that is its image under R. The evaluation $f_V\colon EV\to FV$ of this natural transformation on an object $V\in\mathcal{R}$ will be represented diagrammatically by drawing a line labelled by V on the right-hand side of the usual string diagram for f:



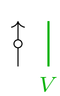
For the remainder of these notes, we fix a $\Bbbk$-linear $\mathcal{H}eis_k$-module category $\mathcal{R}$ that is *locally finite abelian*. This means that

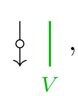
- all objects are of finite length, and
- the space of morphisms between any two objects is finite-dimensional.

Special cases of locally finite abelian categories include *finite abelian categories*, that is, categories equivalent to the category of finite-dimensional modules for a finite-dimensional algebra. In fact, with minor modifications, we can also work with *schurian categories*; see [BSW20a, §2.2]. However, we stick to the case of locally finite abelian module categories to simplify the exposition. Our goal is to show that $\mathcal{R}$ can be given the structure of a Kac–Moody 2-representation.

5.1. Eigenfunctors.

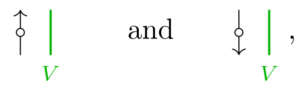
The endofunctors E and F of $\mathcal{R}$ defined by the generating objects $\uparrow$ and $\downarrow$ of $\mathcal{H}eis_k$ are biadjoint, with adjunctions (E,F) and (F,E) defined by the rightwards cups/caps and the leftwards cups/caps, respectively. For $i\in\Bbbk$, let E_i and F_i be the subfunctors of E and F defined on $V\in\mathcal{R}$ by declaring that E_iV and F_iV are the generalized i-eigenspaces of the endomorphisms



and


respectively.

Let us spell this definition out in more detail. Under our assumptions on $\mathcal{R}$, the endomorphism algebras $\mathrm{End}_{\mathcal{R}}(EV)$ and $\mathrm{End}_{\mathcal{R}}(FV)$ are finite dimensional. So, we can define $m_V(u), n_V(u) \in \Bbbk[u]$ to be the (monic) *minimal polynomials* of the endomorphisms



respectively. Then there are injective homomorphisms

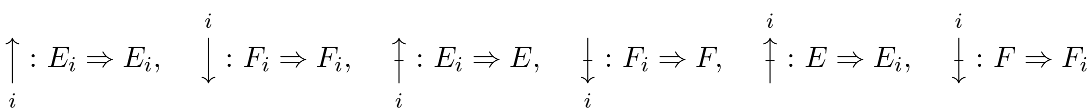
(5.1) $$\Bbbk[u]/(m_V(u)) \hookrightarrow \mathrm{End}_{\mathcal{R}}(EV), \quad \Bbbk[u]/(n_V(u)) \hookrightarrow \mathrm{End}_{\mathcal{R}}(FV),$$

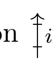
Also let $\epsilon_i(V)$ and $\phi_i(V)$ denote the multiplicities of $i \in \Bbbk$ as a root of the polynomials $m_V(u)$ and $n_V(u)$, respectively. By the Chinese Remainder Theorem, we have that

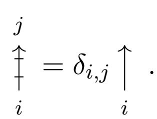
(5.2) $$\Bbbk[u]/(m_V(u)) \cong \bigoplus_{i \in \Bbbk} \Bbbk[u]/\big((u-i)^{\epsilon_i(V)}\big), \qquad \Bbbk[u]/(n_V(u)) \cong \bigoplus_{i \in \Bbbk} \Bbbk[u]/\big((u-i)^{\phi_i(V)}\big).$$

There are corresponding decompositions $1 = \sum_{i \in \Bbbk} e_i$ and $1 = \sum_{i \in \Bbbk} f_i$ of the identity elements of these algebras as a sum of mutually orthogonal idempotents. We define E_iV and F_iV to be the summands of EV and FV, respectively, defined by the images of the idempotents e_i and f_i under (5.1).

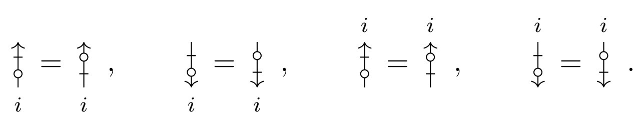
We will represent the identity endomorphisms of the functors E_i and F_i by vertical strings colored by i; see the first pair of diagrams below. The inclusions $E_i \hookrightarrow E$ and $F_i \hookrightarrow F$ are depicted by the second pair of diagrams below. The projections $E \twoheadrightarrow E_i$ and $F \twoheadrightarrow F_i$ are the final pair.



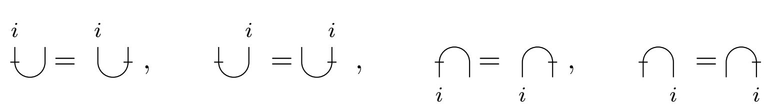
To illustrate the notation, the natural transformation  $: E \Rightarrow E$ is the projection of E onto its summand E_i, while



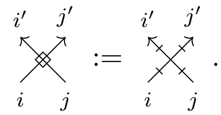
It is also clear from the definition that the endomorphisms of E and F defined by the dots restrict to endomorphisms of the summands E_i and F_i. Representing these restrictions simply by drawing the dots on a string colored by i, we have that

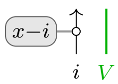
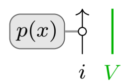


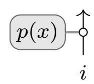
Since the downwards dot is both the left and right mate of the upwards dot, the adjunctions (E, F) and (F, E) induce adjunctions (E_i, F_i) and (F_i, E_i) for all $i \in \Bbbk$. We draw the units and counits of these adjunctions using cups and caps colored by i. Again, the various inclusions and projections commute with these morphisms: for all orientations of the strands, we have



For $i, j, i', j' \in \Bbbk$ we define



Now we come to an extremely useful diagrammatic convention. On any $V \in \mathcal{R}$, the endomorphism  is nilpotent, hence, the notation  makes sense for power series $p(x) \in \Bbbk[\![x - i]\!]$ rather than merely for polynomials. It follows that there is a well-defined natural transformation

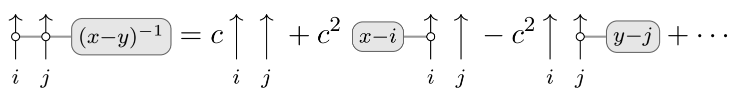
 $: E_i \Rightarrow E_i$

for any $i \in \Bbbk$ and any $p(x) \in \Bbbk[\![x - i]\!]$. The same definition can be made for dots on downward strings too.

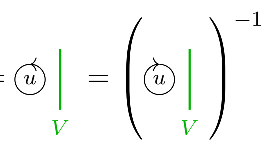
We may do the same for power series in several variables. To give an example, suppose $i \neq j$. Set $c := (i - j)^{-1}$ so that $(x - y)^{-1} \in \Bbbk[\![x - i, y - j]\!]$ has power series expansion

$$c + c^2(x - i) - c^2(y - j) + \text{ (higher order terms)}.$$

Then we have defined the natural transformation



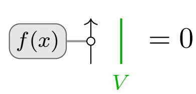
5.2. **Bubbles and central characters.** Any dotted bubble in $\mathcal{H}eis_k$ defines an endomorphism of the identity functor $\mathrm{Id}_{\mathcal{R}}$, i.e., an element of the center of the category $\mathcal{R}$. In particular, for $V \in \mathcal{R}$, dotted bubbles evaluate to elements of the center Z_V of the endomorphism algebra $\mathrm{End}_{\mathcal{R}}(V)$. It is convenient to work with all of these endomorphisms at once in terms of the generating function

(5.3) $\mathbb{O}_V(u) :=$ 

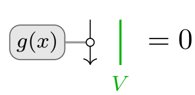
Recalling (4.22), we have $\mathbb{O}_V(u) \in u^k + u^{k-1} Z_V[\![u^{-1}]\!]$.

Lemma 5.1. *Let $V \in \mathcal{R}$ be any object.*

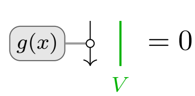
(a) If $f(u) \in Z_V[u]$ is a monic polynomial such that



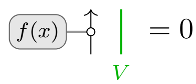
then $g(u) := \mathbb{O}_V(u) f(u)$ is a monic polynomial in $Z_V[u]$ of degree $\deg f(u) + k$ *such that*



(b) If $g(u) \in Z_V[u]$ is a monic polynomial such that

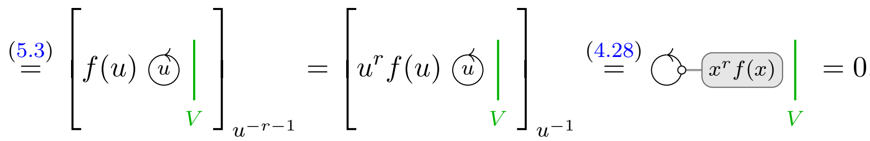


then $f(u) := \mathbb{O}_V(u)^{-1} g(u)$ *is a monic polynomial in* $Z_V[u]$ *of degree* $\deg g(u) - k$ *such that*

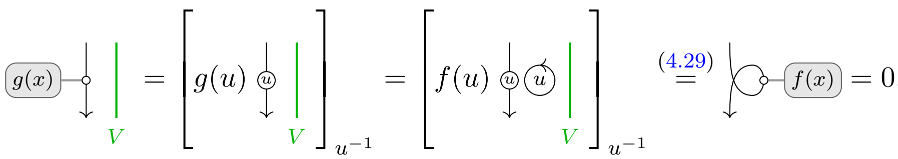


Proof. We prove (a) in the case where V is irreducible, so that $Z_V = \Bbbk$. To show that $g(u)$ is a polynomial, we must show that $[g(u)]_{u^{-r-1}} = 0$ for $r \geq 0$. We compute

$$\left[g(u)\right]_{u^{-r-1}} = \left[\mathbb{O}_V(u) f(u)\right]_{u^{-r-1}}$$



Next, we compute



If $L \in \mathcal{R}$ is irreducible then of course $\mathbb{O}_L(u) \in \Bbbk(\!(u^{-1})\!)$. The following relates the central character information encoded in this generating function to the minimal polynomials $m_L(u)$ and $n_L(u)$ introduced earlier.

Lemma 5.2. *For an irreducible object* $L \in \mathcal{R}$, *we have that*

$$\mathbb{O}_L(u) = \frac{n_L(u)}{m_L(u)}.$$

Proof. Applying Lemma 5.1(a) with $f(u) = m_L(u)$ show that $\mathbb{O}_L(u) m_L(u)$ is a monic polynomial of degree $\deg m_L(u) + k$ that is divisible by $n_L(u)$. Hence,

$$\deg n_L(u) \leq \deg m_L(u) + k.$$

Applying Lemma 5.1(b) with $g(u) = n_L(u)$ shows that $\mathbb{O}_L(u)^{-1} n_L(u)$ is a monic polynomial of degree $\deg n_L(u) - k$ that is divisible by $m_L(u)$. Hence

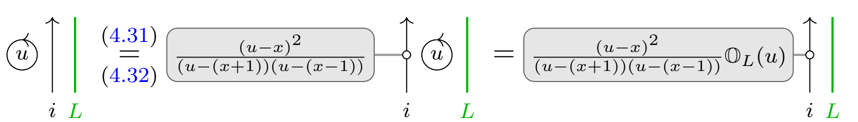
$$\deg m_L(u) \leq \deg n_L(u) - k.$$

We deduce that both inequalities are equalities, and we actually have that $n_L(u) = \mathbb{O}_L(u) m_L(u)$, as desired. □

Lemma 5.3. *Suppose that* $L \in \mathcal{R}$ *is an irreducible object and let* K *be an irreducible subquotient of* $E_i L$ *for some* $i \in \Bbbk$. *Then*

$$\mathbb{O}_K(u) = \frac{(u-i)^2}{(u-(i+1))(u-(i-1))} \mathbb{O}_L(u). \tag{5.4}$$

Proof. This follows from the bubble slide relation (4.31). We have



When we pass to the irreducible subquotient K of $E_i L$, we can replace the occurrences of x in the expression on the right-hand side above with i, and the lemma follows. □

Remark 5.4. The equation (5.4) underpins much of the connection to Kac–Moody algebras. It says that $\mathbb{O}_K(u)$ is obtained from $\mathbb{O}_L(u)$ by increasing the multiplicity of i as a root/pole by 2, and decreasing the multiplicities of $i+1$ and $i-1$ by 1. This should remind the reader of the root system of type A_∞.

Now we define the *spectrum* I of $\mathcal{R}$ to be the union of the sets of roots of the minimal polynomials $m_L(u)$ for all irreducible $L \in \mathcal{R}$. Noting that i is a root of $m_L(u)$ if and only if $E_i L \neq 0$, we have equivalently that I is the set of all $i \in \Bbbk$ such that $E_i L \neq 0$ for some irreducible $L \in \mathcal{R}$. In view of the exactness of E_i, we can drop the word "irreducible" in this characterization: the spectrum I is the set of all $i \in \Bbbk$ such that E_i is a nonzero endofunctor of $\mathcal{R}$. By adjunction, it follows that I is the set of all $i \in \Bbbk$ such that the endofunctor F_i is nonzero, hence, I could also be defined as the union of the sets of roots of the polynomials $n_L(u)$ for all irreducible $L \in \mathcal{R}$. This discussion shows that

$$E = \bigoplus_{i \in I} E_i, \qquad F = \bigoplus_{i \in I} F_i,$$

with each of the endofunctors E_i and F_i written here being nonzero.

Lemma 5.5. *We have that $i \in I$ if and only if $i+1 \in I$.*

Define a_{ij}, $i, j \in I$, by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $(a_{i,j})_{i,j \in I}$ is a Cartan matrix associated to a Dynkin diagram with components of type A_∞.

We fix a complex vector space $\mathfrak{h}$ and linearly independent subsets $\{\alpha_i : i \in I\} \subseteq \mathfrak{h}^*$ and $\{h_i : i \in I\} \subseteq \mathfrak{h}$ such that $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$. Let $X := \{\lambda \in \mathfrak{h}^* : \langle h_i, \lambda \rangle \in \mathbb{Z} \text{ for all } i \in I\}$ be the *weight lattice* and $Y := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ be the *root lattice*. Let $\mathfrak{g}$ be the *Kac–Moody algebra* associated to these data, with Chevalley generators $\{e_i, f_i, h_i : i \in I\}$ and Cartan subalgebra $\mathfrak{h}$. Let ω_i, $i \in I$, be the fundamental weights.

For an irreducible object $L \in \mathcal{R}$, let

$$\text{wt}(L) := \sum_{i \in I} (\phi_i(L) - \epsilon_i(L))\omega_i \in X. \tag{5.5}$$

In other words, due to the definition preceeding (5.2) and Lemma 5.2, $\langle h_i, \text{wt}(L) \rangle \in \mathbb{Z}$ is the multiplicity of $u = i$ as a zero or pole of the rational function $\mathbb{O}_L(u) \in \Bbbk(u)$ for each $i \in I$. Then for $\lambda \in X$ we let $\mathcal{R}_\lambda$ be the Serre subcategory of $\mathcal{R}$ consisting of the objects V such that every irreducible subquotient L of V satisfies $\text{wt}(L) = \lambda$. The point of this definition is that irreducible objects $K, L \in \mathcal{R}$ with $\text{wt}(K) \neq \text{wt}(L)$ have different central characters. Using also the general theory of blocks, it follows that

$$\mathcal{R} = \bigoplus_{\lambda \in X} \mathcal{R}_\lambda. \tag{5.6}$$

We refer to this as the *weight space* decomposition of $\mathcal{R}$.

Lemma 5.6. *For $\lambda \in X$ and $i \in I$, the restrictions of E_i and F_i to $\mathcal{R}_\lambda$ give functors*

$$E_i|_{\mathcal{R}_\lambda} : \mathcal{R}_\lambda \to \mathcal{R}_{\lambda+\alpha_i}, \qquad F_i|_{\mathcal{R}_\lambda} : \mathcal{R}_\lambda \to \mathcal{R}_{\lambda-\alpha_i}.$$

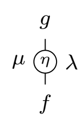
Proof. For E_i, this follows from Lemma 5.3. Then it follows for F_i by adjunction. □

6. The Kac–Moody 2-category

In this section, we introduce our second main category of interest: the Kac–Moody 2-category. This was defined by Khovanov and Lauda [KL10] and Rouquier [Rou08]. In fact, there is such a category associated to any symmetrizable Cartan matrix, but in this paper we are only interested in the ones of Cartan type A_∞, so we specialize to that right away. Our exposition is based on [Bru16], which unified the different approaches of Khovanov–Lauda and Rouquier, and [BD17, §3], which incorporated some renormalizations of the bubbles following the idea of [BHLW16] in order to make the strictly pivotal structure apparent.

6.1. **2-categories.** A strict $\Bbbk$-linear 2-category $\mathfrak{C}$ is a category enriched in the monoidal category of $\Bbbk$-linear categories. This means that, for every two objects A and B, we have a *category* $\mathrm{hom}(A, B)$. The objects of these hom-categories are the 1-morphisms of $\mathfrak{C}$, while the morphisms of the hom-categories are the 2-morphisms of $\mathfrak{C}$.

We will use a string diagram calculus for 2-categories that is very similar to the one we have used for monoidal categories. (In fact, a 2-category with one object is equivalent to a monoidal category!) The difference is that the string diagrams now represent 2-morphisms, and regions are labelled by objects of the 2-category. For example, a 2-morphism $\eta\colon f \to g$ between 1-morphisms $f, g\colon \lambda \to \mu$ is depicted as follows:



.

Note that the domain, λ, of f and g is written on the *right*.

6.2. **Definition of the Kac–Moody 2-category.** Recall that $(a_{ij})_{i,j\in I}$, is a Cartan matrix associated to a Dynkin diagram with components of type A_∞. For $\lambda \in X$, we define

$$\lambda_i = \langle h_i, \lambda \rangle,$$

so that

$$\lambda = \sum_{i \in I} \lambda_i \omega_i.$$

We choose signs $\{\sigma_i(\lambda) : \lambda \in X,\ i \in I\}$ such that

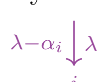
$$\sigma_i(\lambda)\sigma_i(\lambda + \alpha_j) = (-1)^{\delta_{i+1,j}} \qquad \text{for each } j \in I.$$

There is a unique choice satisfying $\sigma_i(\lambda) = 1$ for each $i \in I$ and each λ lying in a set of X/Y-coset representatives.

Definition 6.1. The *Kac–Moody 2-category* is a strict $\Bbbk$-linear 2-category $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$ whose objects are the elements of X. The 1-morphisms are generated by

$$E_i 1_\lambda \colon \lambda \to \lambda + \alpha_i, \qquad F_i 1_\lambda \colon \lambda \to \lambda - \alpha_i, \qquad i \in I,\ \lambda \in X.$$

The identity 2-morphisms of $E_i 1_\lambda$ and $F_i 1_\lambda$ are denoted by

and


respectively. The generating 2-morphisms of $\mathfrak{U}$ are

(6.1) 

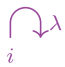
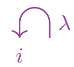
$: E_i 1_\lambda \to E_i 1_\lambda,$ 

$: E_j E_i 1_\lambda \to E_i E_j 1_\lambda,$

(6.2) 

$: 1_\lambda \to F_i E_i 1_\lambda,$ 

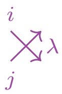
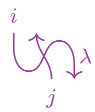
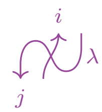
$: 1_\lambda \to E_i F_i 1_\lambda,$

(6.3)  $: E_iF_i1_\lambda \to 1_\lambda$,  $: F_iE_i1_\lambda \to 1_\lambda$,

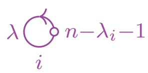
which we refer to as a *dot*, *upward crossing*, *right cup*, *left cup*, *right cap*, and *left cup* respectively. We denote the n-th power of the dot (under vertical composition) by

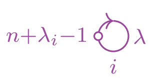
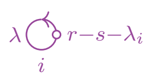
 $: E_i1_\lambda \to E_i1_\lambda$.

We then define the *right crossing* and *left crossing* by

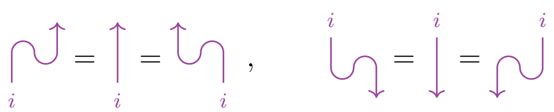
(6.4)  :=  ,  :=  ,

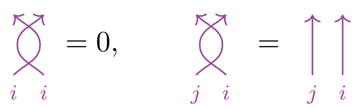
and *negatively dotted bubbles* by

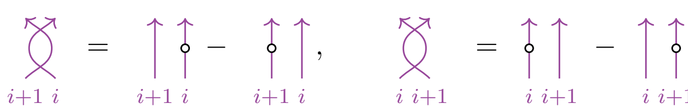
 := $\begin{cases} (-1)^n\sigma_i(\lambda)^{n+1}\det\left(\right.$  $\left.\right)_{r,s=1,\dots,n} & \text{if } \lambda_i \geq n > 0, \\ \sigma_i(\lambda)1_{1_\lambda} & \text{if } \lambda_i \geq n = 0, \\ 0 & \text{if } \lambda_i \geq n < 0, \end{cases}$

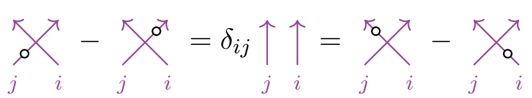
 := $\begin{cases} (-1)^n\sigma_i(\lambda)\det\left(\right.$  $\left.\right)_{r,s=1,\dots,n} & \text{if } -\lambda_i \geq n > 0, \\ \sigma_i(\lambda)1_{1_\lambda} & \text{if } -\lambda_i \geq n = 0, \\ 0 & \text{if } -\lambda_i \geq n < 0. \end{cases}$

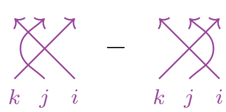
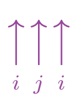
The generating 2-morphisms are subject to the following relations (where we omit the region label when the relation does not depend on it):

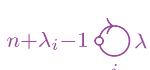
(6.5) 

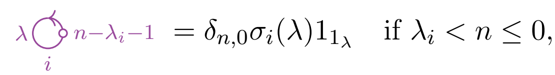
(6.6)  if $|i-j|>1$,

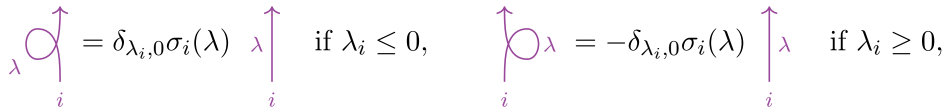
(6.7) 

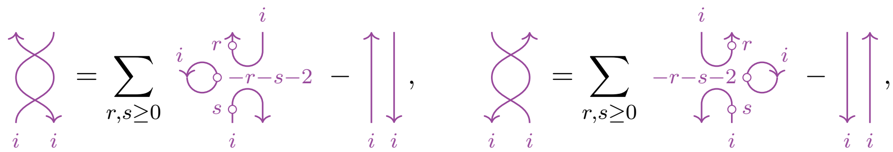
(6.8) 

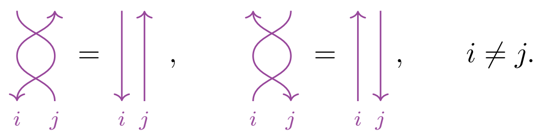
(6.9)  = $\begin{cases}$  $& \text{if } k=i=j-1, \\ -$  $& \text{if } k=i=j+1, \\ 0 & \text{otherwise,} \end{cases}$

(6.10)  $= \delta_{n,0}\sigma_i(\lambda)1_{1_\lambda}$ if $-\lambda_i < n \leq 0$,

(6.11)


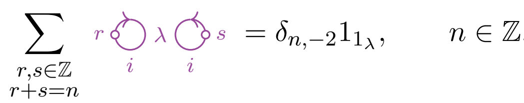
(6.12)


(6.13)


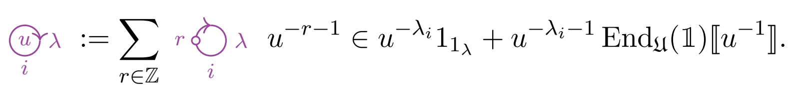
(6.14)


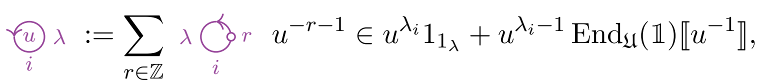
This completes the definition of $\mathfrak{U}(\mathfrak{g})$.

One can show that we have the *infinite grassmannian relation*

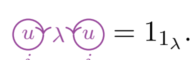
(6.15)


If we introduce the generating functions

(6.16)


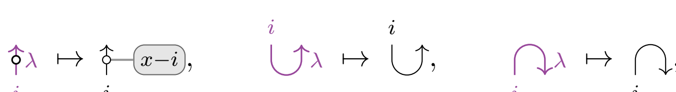
(6.17)


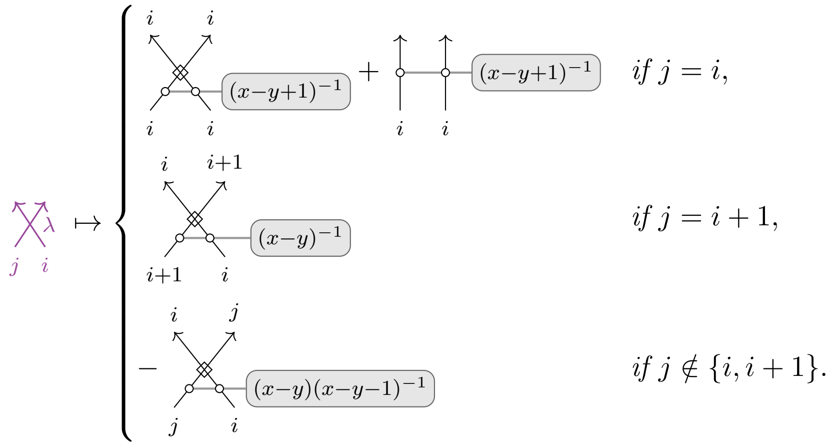
then (6.15) becomes

(6.18)


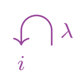
6.3. **Heisenberg to Kac–Moody.** Now we can state the main theorem, which shows how we obtain a 2-representation of the Kac–Moody 2-category from module category of the Heisenberg category. Let $\mathfrak{Cat}$ denote the strict $\Bbbk$-linear 2-category of $\Bbbk$-linear categories.

Theorem 6.2 ([BSW20a, Th. 4.11])**.** *Associated to $\mathcal{R}$, there is a unique 2-representation $\mathbf{R}\colon \mathfrak{U}(\mathfrak{g}) \to \mathfrak{Cat}$ defined on objects by $\lambda \mapsto \mathcal{R}_\lambda$, on generating 1-morphisms by $E_i 1_\lambda \mapsto E_i|_{\mathcal{R}_\lambda}$ and $F_i 1_\lambda \mapsto F_i|_{\mathcal{R}_\lambda}$, and on generating 2-morphisms by*





Remark 6.3. The images of the generating morphisms

 and 

under $\mathbf{R}$ are uniquely determined by the images of the other generators. However, the explicit formulae for these are quite complicated.

7. Final Remarks

One also has a reversed version of Theorem 6.2. Namely, given a 2-representation of $\mathfrak{U}(\mathfrak{g})$ satisfying certain natural assumptions, one can construct a $\mathcal{H}eis_k$-module category. In fact, one can construct an isomorphism between *generalized cyclotomic quotients* of Heisenberg categories and Kac–Moody 2-categories.

We can work over more general fields $\Bbbk$. If $\Bbbk$ has characteristic $p \geq 2$, then $\mathfrak{g}$ is a Kac–Moody algebra whose Dynkin diagram has components of type $A_{p-1}^{(1)}$.

There is also a *quantum Heisenberg category*; see [LS13, BSW20b]. All of the results presented above can be carried out with the Heisenberg category replaced by the quantum Heisenberg category; see [BSW20a] for details.

Examples of categories that admit actions of the Heisenberg category or the quantum Heisenberg category (and hence actions of a Kac–Moody 2-category) for $k \neq 0$ include:

- representations of symmetric groups or Iwahori–Hecke algebras of type A,
- higher level cyclotomic quotients of (degenerate or quantum) affine Hecke algebras.

In these cases, the endofunctors E and F are given by induction and restriction. These modules categories categorify (higher level) Fock space representations of the Heisenberg algebra.

For $k = 0$, the Heisenberg category $\mathcal{H}eis_0$ acts on:

- rational representations of the algebraic group GL_n over $\Bbbk$,
- representations of $\mathfrak{gl}_n(\mathbb{C})$ in the BGG category $\mathcal{O}$,
- analogous categories for the general linear supergroup $\mathrm{GL}_{m|n}$ and its Lie superalgebra,
- finite-dimensional representations of restricted enveloping algebras arising from the Lie algebra $\mathfrak{gl}_n(\Bbbk)$ over a field of positive characteristic,
- analogous categories for the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$, including situations in which q is a root of unity.

In these cases, the endofunctors E and F are given by tensoring with the natural representation and its dual. The dot arises from the action of the Casimir tensor, and the crossing is given by the tensor flip in the degenerate setting, or by its braided analogue defined from the R-matrix in the quantum case.

References

[BD17] J. Brundan and N. Davidson. Categorical actions and crystals. In *Categorification and higher representation theory*, volume 683 of *Contemp. Math.*, pages 105–147. Amer. Math. Soc., Providence, RI, 2017. arXiv:1603.08938, doi:10.1090/conm/683.

[BHLW16] A. Beliakova, K. Habiro, Aaron D. Lauda, and B. Webster. Cyclicity for categorified quantum groups. *J. Algebra*, 452:118–132, 2016. arXiv:1506.04671, doi:10.1016/j.jalgebra.2015.11.041.

[Bru16] J. Brundan. On the definition of Kac-Moody 2-category. *Math. Ann.*, 364(1-2):353–372, 2016. arXiv:1501.00350, doi:10.1007/s00208-015-1207-y.

[Bru18] J. Brundan. On the definition of Heisenberg category. *Algebr. Comb.*, 1(4):523–544, 2018. arXiv:1709.06589, doi:10.5802/alco.

[BSW20a] J. Brundan, A. Savage, and B. Webster. Heisenberg and Kac-Moody categorification. *Selecta Math. (N.S.)*, 26(5):Paper No. 74, 62, 2020. arXiv:1907.11988, doi:10.1007/s00029-020-00602-5.

[BSW20b] J. Brundan, A. Savage, and B. Webster. On the definition of quantum Heisenberg category. *Algebra Number Theory*, 14(2):275–321, 2020. arXiv:1812.04779, doi:10.2140/ant.2020.14.275.

[Kas95] C. Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. doi:10.1007/978-1-4612-0783-2.

[Kho14] M. Khovanov. Heisenberg algebra and a graphical calculus. *Fund. Math.*, 225(1):169–210, 2014. arXiv:1009.3295, doi:10.4064/fm225-1-8.

[KL10] M. Khovanov and A. D. Lauda. A categorification of quantum sl(n). *Quantum Topol.*, 1(1):1–92, 2010. arXiv:0807.3250, doi:10.4171/QT/1.

[LS13] A. Licata and A. Savage. Hecke algebras, finite general linear groups, and Heisenberg categorification. *Quantum Topol.*, 4(2):125–185, 2013. arXiv:1101.0420, doi:10.4171/QT/37.

[Mac98] S. MacLane. *Categories for the working mathematician*. Springer-Verlag, New York-Berlin, second edition, 1998. Graduate Texts in Mathematics, Vol. 5.

[MS18] M. Mackaay and A. Savage. Degenerate cyclotomic Hecke algebras and higher level Heisenberg categorification. *J. Algebra*, 505:150–193, 2018. arXiv:1705.03066, doi:10.1016/j.jalgebra.2018.03.004.

[Pen71] R. Penrose. Applications of negative dimensional tensors. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, pages 221–244. Academic Press, London, 1971.

[Rou08] R. Rouquier. 2-Kac–Moody algebras. 2008. arXiv:0812.5023.

[Sch01] P. Schauenburg. Turning monoidal categories into strict ones. *New York J. Math.*, 7:257–265, 2001. URL: http://nyjm.albany.edu:8000/j/2001/7_257.html.

[TV17] V. Turaev and A. Virelizier. *Monoidal categories and topological field theory*, volume 322 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, 2017. doi:10.1007/978-3-319-49834-8.

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