FROBENIUS 0-HECKE ALGEBRAS

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Abstract. To any symmetric Frobenius superalgebra A we associate a tower of *Frobenius 0-Hecke algebras*. When A is the ground ring, we recover the classical 0-Hecke algebra. We prove a basis theorem for each of these algebras when the parity of the trace map is even.

Contents

1. Introduction

Hecke algebras are q-deformations of the group algebra of the symmetric group. By setting $q = 0$ in the quadratic relation between generators, one obtains the 0-Hecke algebras. The 0-Hecke algebras act naturally on polynomial rings via the Demazure operators, and on $\mathbb{Z}^n$ via the so-called 'bubble-sorting' operators. In [Hua13], Huang also describes the actions of the 0-Hecke algebra on flag varieties and Stanley-Reisner rings. In [KT97], Krob and Thibon showed that their representations can be used to describe the noncommutative and quasi-symmetric functions. Building on these results, Bergeron, Hivert and Thibon defined the 0-Hecke-Clifford algebras to study the peak algebra of symmetric groups [BHT04].

Recently, the affine Hecke algebras, their degenerate analogues and the nilHecke algebras have been generalized in [RS20, Sav20, SS21]. These algebras were augmented with a Frobenius superalgebra A, yielding their classical counterparts when A is the ground ring $\Bbbk$. This approached unified many different generalizations of the algebras found in the literature. Each of these form *towers of algebras* which can be efficiently described using the language of monoidal supercategories.

The goal of the current paper is to initiate this process of 'Frobenization' in the context of the 0-Hecke algebras. Specifically, to each Frobenius superalgebra A and for each positive integer n, we associate a *Frobenius 0-Hecke algebra* $\mathrm{Hecke}_n^0(A)$. To do so, we first construct a strict monoidal, $\Bbbk$-linear supercategory $\mathcal{H}ecke^0(A)$ with one generating object I subject to certain relations. Section 2 and Section 3 provide the necessary background material on monoidal supercatories, towers of algebras and Frobenius superalgebras. These sections are mostly reproduced from [SS21] with permission of the author. In Section 4, we define $\mathrm{Hecke}_n^0(A)$ as the endomorphism algebra of $\mathsf{I}^{\otimes n}$ in $\mathcal{H}ecke^0(A)$. Then we prove a basis theorem for the classical 0-Hecke algebra $\mathrm{Hecke}_n^0(\Bbbk)$. While this result is known to experts, we were unable to find a proof in the literature. We then generalize to a basis theorem for any given Frobenius superalgebra A with an even trace map. This leads naturally to an isomorphism of $\Bbbk$-modules between $\mathcal{H}ecke^0(A)$ and $A^{\otimes n} \otimes \mathrm{Hecke}_n^0(\Bbbk)$.

2. Monoidal supercategories and towers of algebras

Throughout the paper, we fix a commutative ground ring $\Bbbk$. All tensor products are over $\Bbbk$ unless otherwise specified. All superalgebras are associative superalgebras over $\Bbbk$ and all (super)categories are $\Bbbk$-linear. For a homogeneous element a in a vector superspace, we let $\bar{a} \in \mathbb{Z}_2$ denote its parity. When we write an equation involving parities of elements, we implicitly assume these elements are homogeneous; we then extend by linearity.

For superalgebras $A = A_{\bar{0}} \oplus A_{\bar{1}}$ and $B = B_{\bar{0}} \oplus B_{\bar{1}}$, multiplication in the superalgebra $A \otimes B$ is defined by

$$(a' \otimes b)(a \otimes b') = (-1)^{\bar{a}\bar{b}} a'a \otimes bb' \tag{2.1}$$

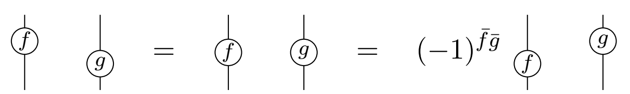
for homogeneous $a, a' \in A$, $b, b' \in B$.

Throughout this paper we will work with *strict monoidal supercategories*, in the sense of [BE17]. We refer the reader to [BSW20, §2] for a summary of this topic well adapted to the current work, or to [BE17] for a thorough treatment. We summarize here a few crucial properties that play an important role in the present paper.

A *supercategory* means a category enriched in the category of vector superspaces with parity-preserving morphisms. Thus, its morphism spaces are vector superspaces and composition is parity-preserving. In a *strict monoidal supercategory*, morphisms satisfy the *super interchange law*:

$$(f' \otimes g) \circ (f \otimes g') = (-1)^{\bar{f}\bar{g}} (f' \circ f) \otimes (g \circ g'). \tag{2.2}$$

We denote the unit object by $\mathbb{1}$ and the identity morphism of an object X by 1_X. We will use the usual calculus of string diagrams, representing the horizontal composition $f \otimes g$ (resp. vertical composition $f \circ g$) of morphisms f and g diagrammatically by drawing f to the left of g (resp. drawing f above g). Care is needed with horizontal levels in such diagrams due to the signs arising from the super interchange law:

(2.3) 

In fact, all of the categories we consider in the current paper will be strict monoidal supercategories generated by a single object I. The domain and codomain of a string diagram can then be read from the number of strands at the bottom and top, respectively, of the diagram. For example,

 $: \mathsf{I}^{\otimes 2} \to \mathsf{I}^{\otimes 2}$ and  $: \mathsf{I} \to \mathsf{I}.$

(In fact, all of our generating morphisms will be endomorphisms, having the same domain and codomain.) For this reason, we will often omit the domain and codomain when introducing the generating morphisms of a category. When numbering strands, we will always number *from right to left.*

A strict monoidal supercategory $\mathcal{C}$ with one generating object I gives rise to a *tower of algebras*

$$\mathrm{End}_{\mathcal{C}}(\mathsf{I}^{\otimes n}), \ n \in \mathbb{N}.$$

We use this idea to introduce various families of algebras in an extremely efficient way, giving a presentation of $\mathcal{C}$ with a small number of generating morphisms and relations. If we then wish to have a presentation of the endomorphism algebras of $\mathcal{C}$ *as superalgebras*, we use the following result.

Proposition 2.1. *Suppose $\mathcal{C}$ is a strict monoidal supercategory with one generating object I and generating morphisms $f_i \in \mathrm{End}_{\mathcal{C}}(\mathsf{I}^{\otimes n_i})$, $i \in I$, subject to the relations $R_j \in \mathrm{End}_{\mathcal{C}}(\mathsf{I}^{\otimes m_j})$, $j \in J$. Then $\mathrm{End}_{\mathcal{C}}(\mathsf{I}^{\otimes n})$ is generated as an algebra by the elements*

$$1_{\mathbb{1}}^{n-n_i-k} \otimes f_i \otimes 1_{\mathbb{1}}^{k}, \quad i \in I, \ 0 \le k \le n - n_i,$$

subject to the relations

$$1_{\mathbb{1}}^{n-m_j-k} \otimes R_j \otimes 1_{\mathbb{1}}^{k}, \quad j \in J,\ 0 \le k \le n-m_j, \tag{2.4}$$

and the relations

$$\left(1_{\mathbb{1}}^{\otimes k_1} \otimes f_i \otimes 1^{\otimes k_2}\right)\left(1_{\mathbb{1}}^{\otimes l_1} \otimes f_j \otimes 1^{\otimes l_2}\right) - (-1)^{\overline{f_i}\,\overline{f_j}}\left(1_{\mathbb{1}}^{\otimes l_1} \otimes f_j \otimes 1^{\otimes l_2}\right)\left(1_{\mathbb{1}}^{\otimes k_1} \otimes f_i \otimes 1^{\otimes k_2}\right) \tag{2.5}$$

for $i, j \in I$, $k_1 + n_i + k_2 = n = l_1 + n_j + l_2$, $k_2 \ge n_j + l_2$.

Proof. This follows from Theorems 5.2 and 5.4 of [Liu18]. Note that the relations (2.5) correspond to the super interchange law. □

Note that the assumptions in Proposition 2.1 are quite strong: all generating morphisms and relations are endomorphisms. All of our categories will have this property.

3. Frobenius superalgebras

We fix a symmetric Frobenius superalgebra A with trace map $\mathrm{tr}\colon A \to \Bbbk$ of parity ε. By definition, this means that tr is a homogeneous $\Bbbk$-linear map of parity ε satisfying

$$\mathrm{tr}(ab) = (-1)^{\bar a \bar b}\,\mathrm{tr}(ba), \quad a, b \in A, \tag{3.1}$$

and A has a basis $\mathbf{B}_A$ with a dual basis $\mathbf{B}_A^\vee = \{b^\vee : b \in \mathbf{B}_A\}$ satisfying

$$\mathrm{tr}(a^\vee b) = \delta_{a,b}, \quad a, b \in \mathbf{B}_A. \tag{3.2}$$

It follows that

$$\sum_{b \in \mathbf{B}_A} \mathrm{tr}(b^\vee a) b = a = \sum_{b \in \mathbf{B}_A} \mathrm{tr}(ab) b^\vee, \quad a \in A. \tag{3.3}$$

Note that $\bar b + \overline{b^\vee} = \varepsilon$. By abuse of notation, we will often refer to A itself as a Frobenius superalgebra, leaving the trace map tr implicit.

The symmetric group $\mathfrak{S}_n$ acts on $A^{\otimes n}$ by superpermutations. In particular, the simple transposition s_i acts by

$$s_i(a_n \otimes \cdots \otimes a_1) = (-1)^{\bar a_i \bar a_{i+1}} a_n \otimes \cdots \otimes a_{i+2} \otimes a_i \otimes a_{i+1} \otimes a_{i-1} \otimes \cdots a_1, \quad a_1, \dots, a_n \in A, \tag{3.4}$$

extended by linearity. Note that here, and throughout the paper, we number factors from *right to left*.

Define

$$\tau := \sum_{b \in \mathbf{B}_A} (-1)^{\varepsilon b} b \otimes b^\vee, \tag{3.5}$$

an element of $A^{\otimes 2}$ which we extend to an element of $A^{\otimes n}$ defined by

$$\tau_i := 1^{\otimes(n-i-2)} \otimes \tau \otimes 1^{\otimes(i-1)}. \tag{3.6}$$

This element of $A^{\otimes n}$ has parity ε and is independent of the chosen basis $\mathbf{B}_A$.

Lemma 3.1. *For each* $b \in \mathbf{B}_A, \mathbf{a} \in A^{\otimes n}$ *and for* $1 \le i \le n-1$, *we have*

$$(b^\vee)^\vee = (-1)^{\bar b + \varepsilon \bar b} b \tag{3.7}$$

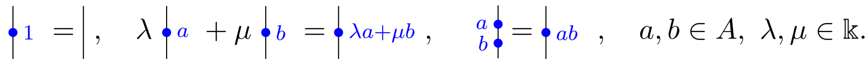
$$\mathbf{a}\tau_i = (-1)^{\varepsilon \bar a} \tau_i s_i(\mathbf{a}). \tag{3.8}$$

Proof. This is a straightforward extension of Lemma 3.2 in [SS21]. □

Definition 3.2. The *Frobenius tower category* $\mathcal{tower}(A)$ is the strict monoidal supercategory with one generating object I, generating morphisms (called *tokens*)

 $a \in A$,

and relations

(3.9) 

We declare the parity of a (token) to be the same as that of a.

Note that relations (3.9) are precisely the relations we need in order to have a homomorphism of superalgebras

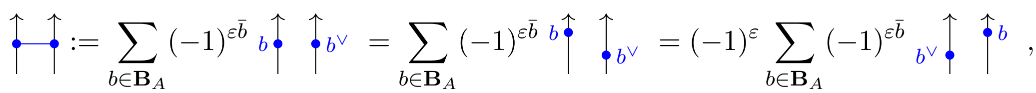
(3.10) $A \to \mathrm{End}_{\mathcal{tower}(A)}(\mathsf{I}), \quad a \mapsto$  $, \ a \in A.$

In fact, $\mathcal{tower}(A)$ is the free monoidal supercategory generated by an object with endomorphism superalgebra A. It follows from Proposition 2.1 that we have an isomorphism

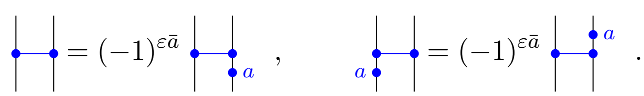
$$A^{\otimes n} \xrightarrow{\cong} \mathrm{End}_{\mathcal{tower}(A)}(\mathsf{I}^{\otimes n}), \tag{3.11}$$

sending $1^{\otimes(n-i)} \otimes a \otimes 1^{\otimes(i-1)}$, $a \in A$, to a token labeled a on the i-th strand. As always, we label strands from *right to left*.

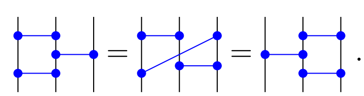
Following [BSW20, §4], we introduce the *teleporters*

(3.12) 

where, in the last equality, we used (3.7) and changed basis in the sum. The teleporter is independent of the chosen basis $\mathbf{B}_A$ and has parity ε. It is the element of $\mathcal{tower}(A)$ corresponding to τ (see (3.5)) under the isomorphism of (3.11). It follows from (3.8) that tokens can "teleport" across teleporters (justifying the terminology) in the sense that, for $a \in A$, we have

(3.13) 

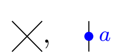
As shown in [MS], the endpoints of teleporters can also teleport. For example, we have

(3.14) 

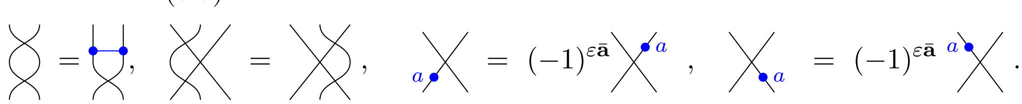
4. Frobenius 0-Hecke algebras

In this section we introduce the tower of Frobenius 0-Hecke algebras.

Definition 4.1. The *Frobenius 0-Hecke category* $\mathcal{Hecke}^0(A)$ is the strict monoidal supercategory with one generating object I and generating morphisms

 $a \in A$,

subject to the relations (3.9) and

(4.1) 

We refer to the generator (crossing) as a *crossing* and declare it to be even. The parity of the token a is the same as the parity of a. For $n \in \mathbb{N}$, we define the *Frobenius 0-Hecke algebra*

$$\mathrm{Hecke}^0_n(A) := \mathrm{End}_{\mathcal{Hecke}^0(A)}(\mathsf{I}^{\otimes n}).$$

Proposition 4.2. *As a superalgebra,* $\mathrm{Hecke}_n^0(A)$ *is isomorphic to the free product of* $A^{\otimes n}$ *and the free associative superalgebra on even generators* $u_1, \dots, u_{n-1}$*, subject to the relations*

$$u_i^2 = \tau_i u_i, \qquad 1 \le i \le n-1, \tag{4.2}$$

$$u_i u_j = u_j u_i, \qquad |i-j| > 1, \tag{4.3}$$

$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}, \qquad 1 \le i \le n-2, \tag{4.4}$$

$$u_i \mathbf{a} = (-1)^{\varepsilon \bar{\mathbf{a}}} s_i(\mathbf{a}) u_i, \qquad 1 \le i \le n-1,\ \mathbf{a} \in A^{\otimes n}. \tag{4.5}$$

Under this isomorphism, u_i *corresponds to the crossing of strands* i *and* $i+1$*,* τ_i *corresponds to the teleporter connecting strands* i *and* $i+1$*, while* $1^{\otimes(n-i)} \otimes a \otimes 1^{\otimes(i-1)}$ *corresponds to a token labeled* a *on strand* i*. As usual, we number strands from right to left.*

Proof. This follows from Proposition 2.1. □

In what follows, we will identify $\mathrm{Hecke}_n^0(A)$ with the algebra presented as in Proposition 4.2.

Remark 4.3. Note that, in fact, the definition of $\mathcal{H}ecke^0(A)$, and hence $\mathrm{Hecke}_n^0(A)$, only involves the superalgebra structure on A, and not the trace map. Thus these are defined for any superalgebra A. The trace map will be important later.

When $A = \Bbbk$, the algebra $\mathrm{Hecke}_n^0(\Bbbk)$ is purely even and $\tau_i = 1^{\otimes n}$, which means we recover the usual 0-Hecke algebra. To construct a basis for $\mathrm{Hecke}_n^0(\Bbbk)$, let $w \in \mathfrak{S}_n$ and define

$$u_w := u_{i_1} u_{i_2} \cdots u_{i_k},$$

where $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for w.

Lemma 4.4. *The element* u_w *is independent of the choice of reduced expression.*

Proof. The relations (4.3) and (4.4) define the *braid relations* on the generators $u_1, \dots, u_{n-1}$. Given two reduced expressions $w = s_{i_1} \dots s_{i_k}$ and $w = s_{i_1} \dots s_{i_l}$, Tits showed in [Tit69] that $s_{i_1} \dots s_{i_l}$ is obtained from $s_{i_1} \dots s_{i_k}$ using only the corresponding braid relations of $\mathfrak{S}_n$. A modern proof of this fact can be found in Björner and Benti, see [BB05, Theorem 3.3.1]. In their paper, the successive applications of the braid relations form a sequence of *braid-moves*. Since we can apply the same sequence of braid-moves on u_w, the independence follows. □

Lemma 4.5. *We have*

$$\ell(s_i w) > \ell(w) \Leftrightarrow w^{-1}(i) < w^{-1}(i+1), \tag{4.6}$$

for each $1 \le i \le n-1$ *and* $w \in \mathfrak{S}_n$.

Proof. We know that

$$\ell(s_i w) > \ell(w) \Leftrightarrow \ell(w^{-1} s_i) > \ell(w^{-1}),$$

since $\ell(s_i w) = \ell(w^{-1} s_i)$ and $\ell(w) = \ell(w^{-1})$. Let $\mathrm{inv}(w)$ denote the number of inversions of a permutation w. Then

$$\ell(w^{-1} s_i) > \ell(w^{-1}) \Leftrightarrow \mathrm{inv}(w^{-1} s_i) > \mathrm{inv}(w^{-1}) \Leftrightarrow w^{-1} s_i(i) > w^{-1}(i),$$

and, since $w^{-1} s_i(i) = w^{-1}(i+1)$, we see that

$$\ell(s_i w) > \ell(w) \Leftrightarrow \ell(w^{-1} s_i) > \ell(w^{-1}) \Leftrightarrow w^{-1}(i+1) > w^{-1}(i),$$

as required. □

Proposition 4.6. *The set* $\{u_w : w \in \mathfrak{S}_n\}$ *forms a basis of* $\mathrm{Hecke}_n^0(\Bbbk)$.

Proof. Let $u \in \text{Hecke}_n^0(\Bbbk)$. By construction, we know that u is a linear combination of products of generators, i.e.

$$u = \sum_{w \in \mathfrak{S}_n} c_w u_w,$$

where $c_w \in \Bbbk$ and $u_w = u_{i_1} \dots u_{i_k}$ is a reduced expression of u_w, for each $w \in \mathfrak{S}_n$. By Lemma 4.4, each u_w is independent of the choice of reduced expression and is indexed by its corresponding element $w = s_{i_1} \dots s_{i_k} \in \mathfrak{S}_n$. Therefore the set $\{u_w : w \in \mathfrak{S}_n\}$ spans $\text{Hecke}_n^0(\Bbbk)$.

To show that $\{u_w : w \in \mathfrak{S}_n\}$ is linearly independent, let

$$V := \bigoplus_{w \in \mathfrak{S}_n} \text{span}\{v_w\}$$

be the $\Bbbk$-module with basis $\{v_w : w \in \mathfrak{S}_n\}$, and consider the map from $\text{Hecke}_n^0(\Bbbk) \times V$ to V defined on the generators by

$$(u_i, v_w) \mapsto u_i v_w = \begin{cases} v_{s_i w} & \text{if } \ell(s_i w) > \ell(w) \\ v_w & \text{if } \ell(s_i w) < \ell(w) \end{cases}, \tag{4.7}$$

for each $1 \leq i \leq n-1, w \in \mathfrak{S}_n$, extended by linearity. We claim this map defines an action of $\text{Hecke}_n^0(\Bbbk)$ on V. To verify this fact, we will show that relations (4.2)–(4.4) are preserved by the map.

Observe that, for each u_i and v_w, we have

$$u_i^2 v_w = \begin{cases} u_i v_{s_i w} & \text{if } \ell(s_i w) > \ell(w) \\ u_i v_w & \text{if } \ell(s_i w) < \ell(w) \end{cases} = \begin{cases} v_{s_i w} & \text{if } \ell(s_i s_i w) < \ell(s_i w) \\ v_w & \text{if } \ell(s_i w) < \ell(w) \end{cases} = u_i v_w,$$

and since $\tau_i = 1^{\otimes n}$, we see that the map preserves relation (4.2).

Now, for each u_i, u_j such that $|i - j| > 1$, we consider the following cases:

(i) Assume $\ell(s_i w), \ell(s_j w) > \ell(w)$. Then, by (4.6), we know that

$$w^{-1}(i) < w^{-1}(i+1) \text{ and } w^{-1}(j) < w^{-1}(j+1).$$

Now, notice that

$$\begin{aligned} \ell(s_j w) > \ell(w) &\Leftrightarrow w^{-1}(j) < w^{-1}(j+1) \\ &\Leftrightarrow w^{-1} s_i(j) < w^{-1} s_i(j+1) && (\text{since } |i-j| > 1) \\ &\Leftrightarrow (s_i w)^{-1}(j) < (s_i w)^{-1}(j+1) \\ &\Leftrightarrow \ell(s_j s_i w) > \ell(s_i w) && (\text{by } (4.6)) \end{aligned}$$

and, by a similar argument,

$$\ell(s_i w) > \ell(w) \Leftrightarrow \ell(s_i s_j w) > \ell(s_j w).$$

Therefore $\ell(s_j s_i w) > \ell(s_i w)$ and $\ell(s_i s_j w) > \ell(s_j w)$. Using (4.7), we have

$$\begin{aligned} u_j u_i v_w &= u_j v_{s_i w} && (\text{since } \ell(s_i w) > \ell(w)) \\ &= v_{s_j s_i w} && (\text{since } \ell(s_j s_i w) > \ell(s_i w)) \\ &= v_{s_i s_j w} && (\text{since } s_i s_j = s_j s_i \text{ when } |i-j| > 1) \\ &= u_i v_{s_j w} && (\text{since } \ell(s_i s_j w) > \ell(s_j w)) \\ &= u_i u_j v_w && (\text{since } \ell(s_j w) > \ell(w)), \end{aligned}$$

therefore (4.3) is satisfied.

(ii) If $\ell(s_iw) < \ell(w) < \ell(s_jw)$, then

$$u_ju_iv_w = u_jv_w = v_{s_jw} = u_iv_{s_jw},$$

where the last equality holds because

$$\ell(s_is_jw) = \ell(s_js_iw) \leq \ell(s_j) + \ell(s_iw) < 1 + \ell(s_jw),$$

which implies $\ell(s_is_jw) < \ell(s_jw)$. Since $u_iv_{s_jw} = u_iu_jv_w$, (4.3) is satisfied.

(iii) If $\ell(s_iw), \ell(s_jw) < \ell(w)$, then

$$u_ju_iv_w = v_w = u_iu_jv_w.$$

These cases confirm that the map preserves relation (4.3). Before verifying (4.4), we need to consider special cases of Lemma 4.5. In particular, notice that

$$\begin{aligned}\ell(s_{i+1}s_iw) > \ell(s_iw) &\Leftrightarrow (s_iw)^{-1}(i+1) < (s_iw)^{-1}(i+2)\\ &\Leftrightarrow w^{-1}s_i(s_i(i)) < w^{-1}s_i(i+2)\\ &\Leftrightarrow w^{-1}(i) < w^{-1}(i+2)\end{aligned}$$

and

$$\begin{aligned}\ell(s_is_{i+1}s_iw) > \ell(s_{i+1}s_iw) &\Leftrightarrow (s_{i+1}s_iw)^{-1}(i) < (s_{i+1}s_iw)^{-1}(i+1)\\ &\Leftrightarrow w^{-1}s_is_{i+1}(i) < w^{-1}s_is_{i+1}(i+1)\\ &\Leftrightarrow w^{-1}(i+1) < w^{-1}(i+2).\end{aligned}$$

Therefore

$$\ell(s_{i+1}s_iw) > \ell(s_iw) \Leftrightarrow w^{-1}(i) < w^{-1}(i+2)$$

and

$$\ell(s_is_{i+1}s_iw) > \ell(s_{i+1}s_iw) \Leftrightarrow w^{-1}(i+1) < w^{-1}(i+2).$$

By similar arguments, we can show that

$$\ell(s_is_{i+1}w) > \ell(s_{i+1}w) \Leftrightarrow w^{-1}(i) < w^{-1}(i+2)$$

and

$$\ell(s_{i+1}s_is_{i+1}w) > \ell(s_is_{i+1}w) \Leftrightarrow w^{-1}(i) < w^{-1}(i+1).$$

To verify (4.4), we now consider the following cases:

(I) If $w^{-1}(i) < w^{-1}(i+1) < w^{-1}(i+2)$, then by (4.6), we have $\ell(s_iw), \ell(s_{i+1}w) > \ell(w)$, $\ell(s_{i+1}s_iw) > \ell(s_iw)$, $\ell(s_is_{i+1}w) > \ell(s_{i+1}w)$, $\ell(s_is_{i+1}s_iw) > \ell(s_{i+1}s_iw)$ and $\ell(s_{i+1}s_is_{i+1}w) > \ell(s_is_{i+1}w)$. Therefore, using (4.7), we have

$$u_iu_{i+1}u_iv_w = v_{s_is_{i+1}s_iw} = v_{s_{i+1}s_is_{i+1}w} = u_{i+1}u_iu_{i+1}v_w,$$

i.e. (4.4) is preserved.

(II) If $w^{-1}(i) < w^{-1}(i+2) < w^{-1}(i+1)$, then by (4.6), we have $\ell(s_iw) > \ell(w) > \ell(s_{i+1}w)$, $\ell(s_{i+1}s_iw) > \ell(s_iw)$, $\ell(s_is_{i+1}w) > \ell(s_{i+1}w)$, $\ell(s_is_{i+1}s_iw) < \ell(s_{i+1}s_iw)$ and $\ell(s_{i+1}s_is_{i+1}w) > \ell(s_is_{i+1}w)$. Therefore, using (4.7), we have

$$u_iu_{i+1}u_iv_w = u_iv_{s_{i+1}s_iw} = v_{s_{i+1}s_iw}$$

and

$$u_{i+1}u_iu_{i+1}v_w = u_{i+1}u_iv_w = u_{i+1}v_{s_iw} = v_{s_{i+1}s_iw},$$

which implies $u_iu_{i+1}u_iv_w = u_{i+1}u_iu_{i+1}v_w$, so (4.4) is preserved.

(III) If $w(i+1) < w(i) < w(i+2)$, then by (4.6), we have $\ell(s_iw) < \ell(w) < \ell(s_{i+1}w)$, $\ell(s_{i+1}s_iw) > \ell(s_iw)$, $\ell(s_is_{i+1}w) > \ell(s_{i+1}w)$, $\ell(s_is_{i+1}s_iw) > \ell(s_{i+1}s_iw)$ and $\ell(s_{i+1}s_is_{i+1}w) < \ell(s_is_{i+1}w)$. Therefore, by (4.7), we have

$$u_iu_{i+1}u_iv_w = u_iu_{i+1}v_w = u_iv_{s_{i+1}w} = v_{s_is_{i+1}w}$$

and

$$u_{i+1}u_iu_{i+1}v_w = u_{i+1}v_{s_is_{i+1}w} = v_{s_is_{i+1}w},$$

which implies $u_iu_{i+1}u_iv_w = u_{i+1}u_iu_{i+1}v_w$, so (4.4) is preserved.

(IV) If $w(i+1) < w(i+2) < w(i)$,then by (4.6), we have $\ell(s_iw) < \ell(w) < \ell(s_{i+1}w)$, $\ell(s_{i+1}s_iw) < \ell(s_iw)$ and $\ell(s_is_{i+1}w) < \ell(s_{i+1}w)$. Therefore, by (4.7), we have

$$u_iu_{i+1}u_iv_w = u_iu_{i+1}v_w = u_iv_{s_{i+1}w} = v_{s_{i+1}w}$$

and

$$u_{i+1}u_iu_{i+1}v_w = u_{i+1}u_iv_{s_{i+1}w} = u_{i+1}v_{s_{i+1}w} = v_{s_{i+1}w},$$

where the last equality holds because $\ell(s_{i+1}w) > \ell(w) \implies \ell(s_{i+1}s_{i+1}w) < \ell(s_{i+1}w)$. This implies $u_iu_{i+1}u_iv_w = u_{i+1}u_iu_{i+1}v_w$, so (4.4) is preserved.

(V) If $w(i+2) < w(i) < w(i+1)$, then by (4.6), we have $\ell(s_{i+1}w) < \ell(w) < \ell(s_iw)$, $\ell(s_{i+1}s_iw) < \ell(s_iw)$ and $\ell(s_is_{i+1}w) < \ell(s_{i+1}w)$. Therefore, by (4.7), we have

$$u_iu_{i+1}u_iv_w = u_iu_{i+1}v_{s_i} = u_iv_{s_iw} = v_{s_iw},$$

where the last equality holds because $\ell(s_iw) > \ell(w) \implies \ell(s_is_iw) < \ell(s_iw)$. Also,

$$u_{i+1}u_iu_{i+1}v_w = u_{i+1}u_iv_w = u_{i+1}v_{s_iw} = v_{s_iw},$$

which implies $u_iu_{i+1}u_iv_w = u_{i+1}u_iu_{i+1}v_w$, so (4.4) is preserved.

(VI) If $w(i+2) < w(i+1) < w(i)$, then by (4.6), we know that $\ell(s_iw), \ell(s_{i+1}w) < \ell(w)$. Therefore

$$u_iu_{i+1}u_iv_w = v_w = u_{i+1}u_iu_{i+1}v_w,$$

i.e. (4.4) is preserved.

These cases confirm that the map preserves relation (4.4). Therefore the map defines an action of $\mathrm{Hecke}_n^0(\Bbbk)$ on V.

Finally, let $v_1 \in V$, where 1 corresponds to the identity in $\mathfrak{S}_n$, and let $\sum_{w\in\mathfrak{S}_n} c_wu_w \in \mathrm{Hecke}_n^0(\Bbbk)$. The action yields

$$\left(\sum_{w\in\mathfrak{S}_n} c_wu_w\right)v_1 = \sum_{w\in\mathfrak{S}_n} c_w(u_wv_1) = \sum_{w\in\mathfrak{S}_n} c_wv_w.$$

Thus, if $\sum\limits_{w\in\mathfrak{S}_n} c_wu_w = 0$, then $\sum\limits_{w\in\mathfrak{S}_n} c_wv_w = 0$. As $\{v_w : w \in \mathfrak{S}_n\}$ forms a basis for V, this implies $c_w = 0$ for each $w \in \mathfrak{S}_n$. Therefore $\{u_w : w \in \mathfrak{S}_n\}$ is also linearly independent and forms a basis of $\mathrm{Hecke}_n^0(\Bbbk)$. □

Now that we have a basis for $\mathrm{Hecke}_n^0(\Bbbk)$, we will extend the result for any symmetric Frobenius superalgebra A whose trace map has even parity. To do so, we require the following lemma.

Lemma 4.7. *We have*

(4.8) $$\tau_i\tau_j = (-1)^{\varepsilon}\tau_j\tau_i, \qquad |i-j| > 1,\ 1 \le i,j \le n-1,$$

(4.9) $$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, \qquad 1 \le i \le n-2.$$

Proof. First, to show (4.8), assume $|i-j|>1$. Then

$$\tau_i\tau_j \overset{(3.8)}{=} (-1)^{\varepsilon^2}\tau_j s_j(\tau_i) = (-1)^{\varepsilon}\tau_j\tau_i,$$

where the last equality holds because $s_j(\tau_i)=\tau_i$ when $|i-j|>1$. To show (4.9), observe that, in the example (3.14), if we label the strands from right to left with i to $i+2$ and extend to elements of $\mathrm{End}_{tower(A)}(\mathsf{I}^{\otimes n})$ by tensoring with unit objects, then the left and right-hand sides are mapped to $\tau_i\tau_{i+1}\tau_i$ and $\tau_{i+1}\tau_i\tau_{i+1}$ under the isomorphism (3.11). Therefore the equality is preserved by the isomorphism of algebras. □

For a given $w=s_{i_1}\dots s_{i_k}\in\mathfrak{S}_n$, we also let $\tau_w := \tau_{i_1}\dots\tau_{i_k}$. Using (4.8) and (4.9), we can show that τ_w is independent of the choice of reduced expression for w, just as we did in Lemma 4.4. By (3.8), we have

$$\mathbf{a}\tau_w = (-1)^{\varepsilon\ell(w)\bar{\mathbf{a}}}\tau_w w^{-1}(\mathbf{a}) \tag{4.10}$$

for each $\mathbf{a}\in A^{\otimes n}$ and $w\in\mathfrak{S}_n$.

Also, for any $\Bbbk$-module V, the action of $\mathfrak{S}_n$ on $A^{\otimes n}$ extends to an action on $A^{\otimes n}\otimes V$ defined by

$$w(\mathbf{a}\otimes v) = w(\mathbf{a})\otimes v, \tag{4.11}$$

for each $w\in\mathfrak{S}_n$ and $\mathbf{a}\otimes v\in A^{\otimes n}\otimes V$, extended by linearity.

Proposition 4.8. *Let $\mathbf{B}_A^{\otimes n}$ be a basis of $A^{\otimes n}$. If $\varepsilon=\bar{0}$, then the set*

$$\mathbf{B}_H := \left\{\mathbf{b}u_w \in \mathrm{Hecke}_n^0(A) : \mathbf{b}\in\mathbf{B}_A^{\otimes n}, w\in\mathfrak{S}_n\right\}$$

forms a basis of $\mathrm{Hecke}_n^0(A)$.

Proof. Let $u\in\mathrm{Hecke}_n^0(A)$ and observe that u can be written

$$u = \sum_{\mathbf{b}\in\mathbf{B}_A^{\otimes n}}\sum_{w\in\mathfrak{S}_n} c_{\mathbf{b},w}\mathbf{b}u_w,$$

with $c_{\mathbf{b},w}\in\Bbbk$ for each $\mathbf{b}\in\mathbf{B}_A^{\otimes n}, w\in\mathfrak{S}_n$, since we may move the terms of $A^{\otimes n}$ to the left of those in $\mathbf{B}_{\mathrm{Hecke}_n^0(\Bbbk)}$ using relation (4.5). Therefore $\mathrm{Hecke}_n^0(A)$ is spanned by $\mathbf{B}_H$.

To show $\mathbf{B}_H$ is linearly independent, let $V=\bigoplus_{w\in\mathfrak{S}_n}\mathrm{span}\{v_w\}$ be the $\Bbbk$-module with basis $\mathbf{B}_V = \{v_w : w\in\mathfrak{S}_n\}$ and consider the action of $\mathrm{Hecke}_n^0(A)$ on $A^{\otimes n}\otimes V$ defined on the generators by

$$\mathbf{a}\cdot(\mathbf{b}\otimes v_w) = \mathbf{ab}\otimes v_w, \tag{4.12}$$

$$u_i\cdot(\mathbf{b}\otimes v_w) = \tau_i\mathbf{b}\otimes u_iv_w, \tag{4.13}$$

for each $1\le i\le n-1, \mathbf{a}\in A^{\otimes n}$ and $\mathbf{b}\otimes v_w\in\mathbf{B}_A^{\otimes n}\otimes\mathbf{B}_V$, extended by linearity. Note that u_iv_w is defined as in (4.7). To show it is indeed an action, we verify that it preserves relations (4.2)–(4.5).

First, for each generator u_i and each basis element $\mathbf{b}\otimes v_w$, we have

$$u_i^2\cdot(\mathbf{b}\otimes v_w) = u_i\cdot(\tau_i\mathbf{b}\otimes u_iv_w) = \tau_i^2\mathbf{b}\otimes u_i^2v_w \overset{(4.7)}{=} \tau_i^2\mathbf{b}\otimes u_iv_w \overset{(4.12)}{=} \tau_iu_i\cdot(\mathbf{b}\otimes v_w).$$

Therefore $u_i^2\cdot(\mathbf{b}\otimes v_w) = \tau_iu_i\cdot(\mathbf{b}\otimes v_w)$, so (4.2) is preserved.

Next, for each u_i, u_j such that $|i-j|>1$ and for each basis element $\mathbf{b}\otimes v_w$, we have

$$\begin{aligned} u_ju_i\cdot(\mathbf{b}\otimes v_w) &= \tau_j\tau_i\mathbf{b}\otimes u_ju_iv_w \\ &= \tau_j\tau_i\mathbf{b}\otimes u_iu_jv_w && \text{(by (4.7))} \\ &= (-1)^{\varepsilon}\tau_i\tau_j\mathbf{b}\otimes u_iu_jv_w && \text{(by (4.8))} \\ &= \tau_i\tau_j\mathbf{b}\otimes u_iu_jv_w && (\text{since } \varepsilon=\bar{0}) \end{aligned}$$

$$= u_i u_j \cdot (\mathbf{b} \otimes v_w),$$

i.e. $u_j u_i \cdot (\mathbf{b} \otimes v_w) = u_i u_j \cdot (\mathbf{b} \otimes v_w)$. Similarly, we have

$$\begin{aligned} u_i u_{i+1} u_i \cdot (\mathbf{b} \otimes v_w) &= \tau_i \tau_{i+1} \tau_i \mathbf{b} \otimes u_i u_{i+1} u_i v_w \\ &= \tau_i \tau_{i+1} \tau_i \mathbf{b} \otimes u_{i+1} u_i u_{i+1} v_w && \text{(by (4.7))} \\ &= \tau_{i+1} \tau_i \tau_{i+1} \mathbf{b} \otimes u_{i+1} u_i u_{i+1} v_w && \text{(by (4.9))} \\ &= u_{i+1} u_i u_{i+1} \cdot (\mathbf{b} \otimes v_w), \end{aligned}$$

for each generator u_i, $1 \leq i \leq n-2$, and for each basis element $\mathbf{b} \otimes v_w$. Therefore relations (4.3) and (4.4) are preserved.

Now let $\mathbf{a} \in A^{\otimes n}$ and observe that, for each generator u_i and basis element $\mathbf{b} \otimes v_w$, we have

$$u_i \mathbf{a} \cdot (\mathbf{b} \otimes v_w) = \tau_i \mathbf{ab} \otimes u_i v_w \overset{(4.11)}{=} (-1)^{\varepsilon \bar{\mathbf{a}}} s_i(\mathbf{a}) \tau_i \mathbf{b} \otimes u_i v_w = (-1)^{\varepsilon \bar{\mathbf{a}}} s_i(\mathbf{a}) u_i \cdot (\mathbf{b} \otimes v_w),$$

therefore relation (4.5) is preserved. Hence we see that (4.12) is an action.

Finally, suppose

$$\sum_{\mathbf{b} \in \mathbf{B}_A^{\otimes n}} \sum_{w \in \mathfrak{S}_n} c_{\mathbf{b},w} \mathbf{b} u_w = 0, \text{ with } c_{\mathbf{b},w} \in \Bbbk \text{ for each } \mathbf{b} \in \mathbf{B}_A^{\otimes n}, w \in \mathfrak{S}_n,$$

and observe that

$$\begin{aligned} \left(\sum_{\mathbf{b} \in \mathbf{B}_A^{\otimes n}} \sum_{w \in \mathfrak{S}_n} c_{\mathbf{b},w} \mathbf{b} u_w \right) \cdot (1 \otimes v_1) &\overset{(4.12)}{=} \sum_{\mathbf{b} \in \mathbf{B}_A^{\otimes n}} \sum_{w \in \mathfrak{S}_n} c_{\mathbf{b},w} (\mathbf{b} \tau_w \otimes v_w) \\ &\overset{(4.10)}{=} \sum_{\mathbf{b} \in \mathbf{B}_A^{\otimes n}} \sum_{w \in \mathfrak{S}_n} (-1)^{\varepsilon \ell(w) \bar{\mathbf{b}}} \tau_w \cdot (w^{-1}(c_{\mathbf{b},w} \mathbf{b}) \otimes v_w) \\ &\overset{(4.10),(4.11)}{=} \sum_{w \in \mathfrak{S}_n} (-1)^{\varepsilon \ell(w) \bar{\mathbf{b}}} \tau_w \cdot \left(\sum_{\mathbf{b} \in \mathbf{B}_A^{\otimes n}} w^{-1}(c_{\mathbf{b},w} \mathbf{b} \otimes v_w) \right) \\ &\overset{(4.11)}{=} \sum_{w \in \mathfrak{S}_n} (-1)^{\varepsilon \ell(w) \bar{\mathbf{b}}} \tau_w \cdot \left(w^{-1} \left(\sum_{\mathbf{b} \in \mathbf{B}_A^{\otimes n}} c_{\mathbf{b},w} (\mathbf{b} \otimes v_w) \right) \right) \\ &= 0. \end{aligned}$$

Therefore $\sum_{\mathbf{b} \in \mathbf{B}_A^{\otimes n}} c_{\mathbf{b},w} (\mathbf{b} \otimes v_w) = 0$ for each $w \in \mathfrak{S}_n$. This implies $c_{\mathbf{b},w} = 0$ for each basis element $\mathbf{b} \otimes v_w$, as $\mathbf{B}_A^{\otimes n} \otimes \mathbf{B}_V$ is linearly independent. Therefore $\{\mathbf{b} u_w : \mathbf{b} \in \mathbf{B}_A^{\otimes n}, w \in \mathfrak{S}_n\}$ is also linearly independent, which makes it a basis for $\mathrm{Hecke}_n^0(A)$. □

Corollary 4.9. *For each A with $\varepsilon = \bar{0}$, we have an isomorphism of $\Bbbk$-modules*

$$\mathrm{Hecke}_n^0(A) \cong A^{\otimes n} \otimes \mathrm{Hecke}_n^0(\Bbbk), \tag{4.14}$$

and the two factors are subalgebras.

Proof. The multiplication $\mathbf{b} \otimes u_w \mapsto \mathbf{b} u_w$ maps basis-to-basis, so it is an isomorphism of $\Bbbk$-modules. □

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