LAMBDA-RINGS

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Abstract. We give a brief introduction to the theory of λ-rings accessible to undergraduate students. We begin by introducing the concepts of tensor product, exterior power, and graded vector spaces. Then we give the definition of λ-rings and discuss several examples. Finally, we explain how direct sums, tensor products, and exterior powers of (graded) vector spaces induce λ-ring structures on the Grothendieck groups of these categories.

Contents

1. Introduction

In algebra, a λ-ring is a commutative ring equipped with a λ-operation. The study of λ-rings is mainly motivated by the behaviour of tensor products and exterior powers over vector spaces. The categories of vector spaces and graded vector spaces, with direct sum, tensor product, and exterior power, induce the addition, multiplication, and λ-operation in λ-rings of integers $\mathbb{Z}$ and Laurent polynomials.

The purpose of this paper is to give a brief introduction to the subject accessible to undergraduate students with basic knowledge of linear algebra and group theory. In Section 2, we provide the definition of the tensor product and show its basic properties, i.e. the universal property, its dimension, and a kind of distributivity. In Section 3, we define the exterior power of vector spaces and prove the theorem of dimensions of exterior powers. After that, we explain graded vector spaces and graded dimension of tensor products and exterior powers in Section 4. Then, in Section 5, the axioms of λ-rings are given as well as some simple examples of λ-rings, including the integers $\mathbb{Z}$, the ring of power series, and the ring of symmetric functions. Finally, we explain how the operations of direct sum, tensor product, and exterior power of (graded) vector spaces, induce λ-ring structures on the integers and on the ring of Laurent polynomials, when we identify these rings with the Grothendieck groups of the categories of finite-dimensional vector spaces and finite-dimensional graded vector spaces.

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2. Tensor Product

Definition 2.1. (Bilinear map). Let U, V, and W be three vector spaces over the same field F. Then a *bilinear map* $b\colon U \times V \to W$ is a mapping such that

(a) $b(u_1 + u_2, v) = b(u_1, v) + b(u_2, v)$ for all $u_1, u_2 \in U$, $v \in V$
(b) $b(u, v_1 + v_2) = b(u, v_1) + b(u, v_2)$ for all $u \in U$, $v_1, v_2 \in V$
(c) $b(\alpha u, v) = \alpha b(u, v) = b(u, \alpha v)$ for all $u \in U$, $v \in V$, $\alpha \in F$

In other words, $b(\cdot, v)$ is linear from U to W for all $v \in V$, and $b(u, \cdot)$ is linear from V to W for all $u \in U$.

Example 2.2. Let M be an $m \times n$ complex matrix, U be the vector space $\mathbb{C}^m$, and V be the vector space $\mathbb{C}^n$. Define the map $b\colon U \times V \to \mathbb{C}$ by $b(u, v) = u^T M v$ for all $u \in U$ and $v \in V$ where u^T is the transpose of vector u. Then, the function b is a bilinear map.

Definition 2.3. (Tensor product). Let U, V be vector spaces. The *tensor product* of U and V, denoted $U \otimes V$, is a vector space together with a bilinear map $\otimes\colon U \times V \to U \otimes V$ such that for any bilinear map $b\colon U \times V \to W$, there exists a unique linear map $\ell\colon U \otimes V \to W$ such that $b = \ell \circ \otimes$, that is, such that the following diagram commutes:

$$\begin{tikzcd} U \times V \arrow[r, "\otimes"] \arrow[dr, "b"'] & U \otimes V \arrow[d, dashed, "\ell"] \\ & W \end{tikzcd}$$

Definition 2.4. (Free vector space). Let S be a set and F be a field. The *free vector space* $F(S)$ generated by S is defined as

$$F(S) = \{f\colon S \to F \mid f(s) = 0 \text{ for all } s \in S \setminus S' \text{ where } S' \text{ is some finite subset of } S\}$$

where the operations on $F(S)$ are defined as pointwise addition and scalar multiplication.

Definition 2.5. (Quotient space). Let V be a vector space and $W \subseteq V$ be a subspace of V. Define $\sim$ to be the equivalence relation such that $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$ for all $v_1, v_2 \in V$. Then, the *quotient space* V/W is defined as $V/\sim$ and the elements in V/W are equivalence classes $[v]$ for $v \in V$.

Definition 2.6. (Tensor product). Let U and V be vector spaces over field F. Define $W \subseteq F(U \times V)$ to be

$$\begin{aligned} W = {} & \{(u_1 + u_2, v) - (u_1, v) - (u_2, v) \mid u_1, u_2 \in U,\ v \in V\} \\ & \cup \{(u, v_1 + v_2) - (u, v_1) - (u, v_2) \mid u \in U,\ v_1, v_2 \in V\} \\ & \cup \{\alpha(u, v) - (\alpha u, v) \mid u \in U,\ v \in V,\ \alpha \in F\} \\ & \cup \{\alpha(u, v) - (u, \alpha v) \mid u \in U,\ v \in V,\ \alpha \in F\}. \end{aligned}$$

Then, the *tensor product* of U and V, denoted $U \otimes V$, is defined as $U \otimes V = F(U \times V)/W$. For $u \in U$ and $v \in V$, the *tensor product* of u and v, denoted $u \otimes v$, is defined as $u \otimes v = [(u, v)]$.

It follows from the definition of the quotient space that the tensor product satisfies the following properties:

(a) $(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v$ for all $u_1, u_2 \in U$ and $v \in V$.
(b) $u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2$ for all $u \in U$ and $v_1, v_2 \in V$.
(c) $\alpha u \otimes v = \alpha(u \otimes v) = u \otimes \alpha v$ for all $u \in U, v \in V$, and $\alpha \in F$

Remark 2.7. One can show that the tensor product as defined in Definition 2.6 satisfies the universal property of Definition 2.3.

Theorem 2.8. *Let U and V be two finite-dimensional vector spaces over field F. Then,*

$$\dim(U \otimes V) = \dim U \dim V.$$

Proof. Let $\dim U = m$ and $\dim V = n$. Then, U has a basis $S_1 = \{u_1, u_2, \ldots, u_m\}$ and V has a basis $S_2 = \{v_1, v_2, \ldots, v_n\}$. Now, define $S = \{u_i \otimes v_j : u_i \in S_1,\ v_j \in S_2\}$. The set S is a subset of $U \otimes V$ with mn elements, so it is enough to prove that S is a basis of $U \otimes V$.

For any element that can be written as $u \otimes v \in U \otimes V$, let $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_m u_m$ and $v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$ for $\alpha_i, \beta_j \in F$. Then, it becomes

$$u \otimes v = (\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_m u_m) \otimes (\beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j (u_i \otimes v_j).$$

Then, any arbitrary element $x = u \otimes v + u' \otimes v' + \ldots$ can be written as

$$x = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} (u_i \otimes v_j).$$

It follows $x \in \mathrm{Span}(S)$, so $U \otimes V \subseteq \mathrm{Span}(S)$. Obviously, $S \subseteq U \otimes V$, so $\mathrm{Span}(S) \subseteq U \otimes V$. Thus, we have $U \otimes V = \mathrm{Span}(S)$.

Let $\psi_1, \ldots, \psi_m$ be the dual basis to S_1 and let $\phi_1, \ldots, \phi_n$ be the dual basis to S_2. Then define the maps $b_{i,j}(u, v) = \psi_i(u)\phi_j(v)$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. When v is fixed, $\phi_j(v)$ becomes a fixed scalar $c \in F$, then $b_{i,j}(u, v) = c\psi_i(u)$. The dual basis $\psi_i(u)$ is a linear transformation, so $b_{i,j}(\cdot, v)$ is linear from U to F for all $v \in V$. Similarly, $b_{i,j}(u, \cdot)$ is linear from V to F for all $u \in U$. Hence, $b_{i,j}(u, v) = \psi_i(u)\phi_j(v)$ are bilinear maps. By the universal property, these induce maps on the tensor product, i.e. for all $1 \leq k \leq m, 1 \leq l \leq n$, there exist $\ell_{k,l}$ such that $\ell_{k,l}(u \otimes v) = b_{k,l}(u, v)$.

Now, fix $1 \leq k \leq m, 1 \leq l \leq n$, suppose

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} u_i \otimes v_j = 0 \text{ for some } c_{i,j} \in F, 1 \leq i \leq m, 1 \leq j \leq n.$$

Then, $\ell_{k,l}(u \otimes v) = \ell_{k,l}(0) = 0$, which means

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} b_{k,l}(u_i, v_j) = c_{k,l} = 0 \text{ for } 1 \leq i \leq m, 1 \leq j \leq n.$$

Thus, $c_{k,l} = 0$ for all $1 \leq k \leq m, 1 \leq l \leq n$. Consequently, S is linearly independent.

Therefore, S is a basis of $U \otimes V$. So, we have $\dim(U \otimes V) = |S| = mn = \dim U \dim V$. $\square$

Proposition 2.9 (Distributive law of tensor product over direct sum)**.** *Let U, V, W be finite-dimensional vector spaces, then $U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W)$.*

Proof. Let U be an l-dimensional vector space, V be an m-dimensional vector space and W be an n-dimensional vector space. Then, by Theorem 2.8, the dimension of $U \otimes (V \oplus W)$ is

$$\begin{aligned}\dim(U \otimes (V \oplus W)) = \dim(U)\dim(V \oplus W) &= \dim(U)(\dim(V) + \dim(W)) \\ &= \dim(U)\dim(V) + \dim(U)\dim(W).\end{aligned}$$

On the other hand, we have

$$\dim((U \otimes V) \oplus (U \otimes W)) = \dim(U \otimes V) + \dim(U \otimes W) = \dim(U)\dim(V) + \dim(U)\dim(W).$$

Thus, $U \otimes (V \oplus W)$ and $(U \otimes V) \oplus (U \otimes W)$ have the same dimension, so they are isomorphic. $\square$

3. Exterior Power

If we generalize the idea of tensor product in Definition 2.3, we can replace the vector spaces U, V with vector spaces $V_1, V_2, \ldots V_k$ and change the bilinear map to multilinear map. Then, we have the tensor product of k vector spaces, $V_1 \otimes V_2 \otimes \ldots \otimes V_k$. Let V be some vector space over field F, define

$$T^0(V) = F,$$
$$T^1(V) = V, \text{ and}$$
$$T^k(V) = V^{\otimes k} = \underbrace{V \otimes V \otimes \ldots V}_{k} \text{ for } k \geq 2.$$

Definition 3.1. (Exterior power). Let $\mathcal{A}^k$ be the subspace of $T^k(V)$ spanned by vectors in

$$S = \{v_1 \otimes v_2 \otimes \ldots \otimes v_k \mid v_1, v_2, \ldots v_k \in V,\ v_i = v_j \text{ for some } i \neq j\}.$$

Then, the k-th *exterior power* of V is the quotient space $\bigwedge^k(V) = T^k(V)/\mathcal{A}^k$. It follows that each element in $\bigwedge^k(V)$ is of the form $[u_1 \otimes u_2 \otimes \ldots \otimes u_k + v_1 \otimes v_2 \otimes \ldots \otimes v_k + \ldots]$, denoted $u_1 \wedge u_2 \wedge \cdots \wedge u_k + v_1 \wedge v_2 \wedge \cdots \wedge v_k + \ldots$ for some $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k, \ldots \in V$.

Definition 3.2. (Alternating multilinear map) Let $f\colon V^k \to W$ be a multilinear map. Then, f is *alternating* if $f(v_1, \ldots, v_k) = 0$ whenever $v_i = v_{i+1}$ for some $1 \leq i \leq n-1$. Then, all alternating multilinear maps have the property $f(v_1, \ldots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$.

Remark 3.3. It follows from the definition that the exterior power satisfies a universal property. Precisely, for any alternating multilinear map $m\colon V^k \to W$, there exists a unique linear map $\ell\colon \bigwedge^k(V) \to W$ such that $m = \ell \circ \bigwedge^k$, that is, such that the following diagram commutes:

$$\begin{tikzcd} V^k \arrow[r, "\wedge^k"] \arrow[dr, "m"'] & \bigwedge^k(V) \arrow[d, dashed, "\ell"] \\ & W \end{tikzcd}$$

Lemma 3.4. *Let V be a vector space, and $v_1, \dots, v_k \in V$. Then,*

$$v_1 \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \wedge \cdots \wedge v_{j-1} \wedge v_i \wedge v_{j+1} \wedge \cdots \wedge v_k$$

for all $1 \leq i < j \leq k$.

Proof. The element

$$\begin{aligned} x &= v_1 \wedge \cdots \wedge v_{i-1} \wedge (v_i - v_j) \wedge v_{i+1} \wedge \cdots \wedge v_{j-1} \wedge (v_i - v_j) \wedge v_{j+1} \wedge \cdots \wedge v_k \\ &= -v_1 \wedge \cdots \wedge v_{i-1} \wedge (v_i - v_j) \wedge v_{i+1} \wedge \cdots \wedge v_{j-1} \wedge (v_j - v_i) \wedge v_{j+1} \wedge \cdots \wedge v_k \end{aligned}$$

has $v_i - v_j$ at positions i and j where $i \neq j$. So, by the definition of $\mathcal{A}^k$, $x = 0$ in $\bigwedge^k(V)$. Then,

$$\begin{aligned} 0 &= (v_1 \wedge \cdots \wedge v_k) - (-v_1 \wedge \cdots \wedge v_{i-1} \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_{j-1} \wedge v_i \wedge v_{j+1} \wedge \cdots \wedge v_k) \\ &\quad - (v_1 \wedge \cdots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \wedge \cdots \wedge v_{j-1} \wedge v_j \wedge v_{j+1} \wedge \cdots \wedge v_k) \\ &\quad + (v_1 \wedge \cdots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \wedge \cdots \wedge v_{j-1} \wedge v_i \wedge v_{j+1} \wedge \cdots \wedge v_k) \\ &= (v_1 \wedge \cdots \wedge v_k) - 0 - 0 + (v_1 \wedge \cdots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \wedge \cdots \wedge v_{j-1} \wedge v_i \wedge v_{j+1} \wedge \cdots \wedge v_k) \\ &= (v_1 \wedge \cdots \wedge v_k) + (v_1 \wedge \cdots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \wedge \cdots \wedge v_{j-1} \wedge v_i \wedge v_{j+1} \wedge \cdots \wedge v_k). \end{aligned}$$

Therefore, $v_1 \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \wedge \cdots \wedge v_{j-1} \wedge v_i \wedge v_{j+1} \wedge \cdots \wedge v_k$. □

Theorem 3.5. *Let V be an n-dimensional vector space with a basis $\{e_1, e_2, \dots, e_n\}$. Then the set $S = \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$ is a basis of $\bigwedge^k(V)$, and* $\dim \bigwedge^k(V) = \binom{n}{k}$.

Proof. For any element that can be written as $v_1 \wedge \cdots \wedge v_k$ where $v_1, \dots, v_k$ are vectors in V, let $v_i = \alpha_{i,1}e_1 + \alpha_{i,2}e_2 + \ldots + \alpha_{i,n}e_n$ for $1 \leq i \leq n$ and $\alpha_i \in F$. Then, v is in span of the set $S' = \{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_j \leq n \text{ for all } 1 \leq j \leq k\}$ and we have $S \subseteq S'$. For any element in $v = e_{i_1} \wedge \cdots \wedge e_{i_k} \in S' \setminus S$, v has either $i_a = i_b$ or $i_a > i_b$ for some $a < b$. The former case leads v to be 0, and the later case can be reduced to $\pm v'$ for some $v' \in S$ by applying Lemma 3.4. So, any element $u_1 \wedge u_2 \wedge \cdots \wedge u_k + v_1 \wedge v_2 \wedge \cdots \wedge v_k + \ldots$ in $\bigwedge^k(V)$ is in $\mathrm{Span}(S') = \mathrm{Span}(S)$. Obviously, $S \subseteq \bigwedge^k(V)$, so $\mathrm{Span}(S) \subseteq \bigwedge^k(V)$. Thus, we have $\bigwedge^k(V) = \mathrm{Span}(S)$.

Let $\psi_1, \dots, \psi_n$ be the dual basis to $\{e_1, e_2, \dots, e_n\}$. We can define the $n \times k$ matrix $A(v_1, \dots, v_k)$ to be $A = [a_{i,j}]$ where $a_{i,j} = \psi_i(v_j)$. Then, we define the maps $m_{i_1,\dots,i_k} : V^k \to F$ for $1 \leq i_1 < \ldots < i_k \leq n$ to be $m_{i_1,\dots,i_k}(v_1, \dots, v_k) = \det B(v_1, \dots, v_k)$ where $B(v_1, \dots, v_k)$ is the $k \times k$ submatrix of $A(v_1, \dots, v_k)$ consisting of the $i_1, \dots, i_k$ rows of $A(v_1, \dots, v_k)$. Then, the maps $m_{i_1,\dots,i_k}$ are multilinear by properties of the determinant. If $v_i = v_{i+1}$ for some $1 \leq i \leq n-1$, then the matrix $A(v_1, \dots, v_k)$ will have the same i-th and $(i+1)$-th column. So, the submatrix $B(v_1, \dots, v_k)$ have two identical columns. Since $B(v_1, \dots, v_k)$ is a singular matrix, it has determinant of 0. Then, we have $m_{i_1,\dots,i_k}(v_1, \dots, v_k) = 0$. Hence, the maps $m_{i_1,\dots,i_k}$ are alternating. By the universal property, these induce maps on the exterior power, i.e. for all $1 \leq i_1 < \ldots < i_k \leq n$, there exist $\ell_{i_1,\dots,i_k}$ such that $\ell_{i_1,\dots,i_k}(v_1 \wedge \cdots \wedge v_k) = m_{i_1,\dots,i_k}(v_1, \dots, v_k)$.

Now, fix $1 \leq j_1 < \ldots < j_k \leq n$, and suppose

$$\sum_{1 \leq i_1 < \ldots < i_k \leq n} c_{i_1,\dots,i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} = 0 \text{ for some } c_{i_1,\dots,i_k} \in F.$$

Then, by the universal property, we have

$$\sum_{1\le i_1<\ldots<i_k\le n} c_{i_1,\ldots,i_k} m_{j_1,\ldots,j_k}(e_{i_1},\ldots,e_{i_k}) = 0.$$

However, $m_{j_1,\ldots,j_k}(e_{i_1},\ldots,e_{i_k}) = 1$ if $i_1 = j_1,\ldots,i_k = j_k$, and $m_{j_1,\ldots,j_k}(e_{i_1},\ldots,e_{i_k}) = 0$ otherwise. Hence,

$$\sum_{1\le i_1<\ldots<i_k\le n} c_{i_1,\ldots,i_k} m_{j_1,\ldots,j_k}(e_{i_1},\ldots,e_{i_k}) = c_{j_1,\ldots,j_k} = 0.$$

Thus, $c_{j_1,\ldots,j_k} = 0$ for all $1 \le j_1 < \ldots < j_k \le n$. So, the set S is linearly independent.

Since $\operatorname{Span}(S) = \bigwedge^k(V)$ and S is linearly independent, we have that S is a basis of $\bigwedge^k(V)$. Since the basis S has cardinality $\binom{n}{k}$, it follows that $\dim \bigwedge^k(V) = \binom{n}{k}$. □

4. Graded Vector Spaces

Definition 4.1. (Graded vector space) A *graded vector space*, is a vector space V together with a decomposition into a direct sum of the form

$$V = \bigoplus_{i\in\mathbb{Z}} V_i.$$

We say that elements of V_i are *homogeneous of degree i*.

Definition 4.2. (Graded dimension) Let V be a graded vector space, then the *graded dimension* of V is defined as

$$\operatorname{grdim} V = \sum_{i\in\mathbb{Z}} (\dim V_i) q^i$$

where q is an indeterminate.

Example 4.3. Let V be an n-dimensional vector space, then $W = \bigwedge^*(V) = \bigoplus_{k\in\mathbb{Z}} \bigwedge^k(V)$ is a graded vector space, and

$$\operatorname{grdim} W = \sum_{i=0}^{n} \binom{n}{i} q^i = (1+q)^n.$$

Example 4.4. Let V be an n-dimensional vector space, then $W = T^*(V) = \bigoplus_{k\in\mathbb{Z}} T^k(V)$ is a graded vector space, and

$$\operatorname{grdim} W = \sum_{k\ge 0} \dim T^k(V) q^k = \sum_{k\ge 0} n^k q^k.$$

Definition 4.5. (Direct sum of graded vector spaces) Let V, W be two graded vector spaces. Then, the *direct sum* of V and W is a graded vector space with the decomposition

$$V \oplus W = \bigoplus_{n\in\mathbb{Z}} (V\oplus W)_n$$

where

$$(V\oplus W)_n = V_n \oplus W_n.$$

Theorem 4.6. *Let V, W be two graded vector spaces with* $\operatorname{grdim} V = f_1(q)$ *and* $\operatorname{grdim} W = f_2(q)$*, then* $\operatorname{grdim}(V\oplus W) = f_1(q) + f_2(q)$.

Proof. Let $f_1(q) = a_0 + a_1 q + a_{-1}q^{-1} + a_2 q^2 + a_{-2}q^{-2} + \dots$ where $a_i = \dim V_i$ for all $i \in \mathbb{Z}$. Let $f_2(q) = b_0 + b_1 q + b_{-1}q^{-1} + b_2 q^2 + b_{-2}q^{-2} + \dots$ where $b_j = \dim W_j$ for all $j \in \mathbb{Z}$. Then, $\dim(V \oplus W)_n = \dim(V_n \oplus W_n) = a_n + b_n$. Thus, $\operatorname{grdim}(V \oplus W) = f_1(q) + f_2(q)$. □

Definition 4.7. (Tensor product of graded vector spaces) Let V, W be two graded vector spaces. Then, the *tensor product* of V and W is a graded vector space with the decomposition

$$V \otimes W = \bigoplus_{n\in\mathbb{Z}} (V \otimes W)_n$$

where

$$(V \otimes W)_n = \bigoplus_{i+j=n} (V_i \otimes W_j).$$

Theorem 4.8. *Let V, W be two graded vector spaces with $\operatorname{grdim} V = f_1(q)$ and $\operatorname{grdim} W = f_2(q)$, then $\operatorname{grdim}(V \otimes W) = f_1(q)f_2(q)$.*

Proof. Let $f_1(q) = a_0 + a_1 q + a_{-1}q^{-1} + a_2 q^2 + a_{-2}q^{-2} + \dots$ where $a_i = \dim V_i$ for all $i \in \mathbb{Z}$. Let $f_2(q) = b_0 + b_1 q + b_{-1}q^{-1} + b_2 q^2 + b_{-2}q^{-2} + \dots$ where $b_j = \dim W_j$ for all $j \in \mathbb{Z}$. Now, we regard $(V \otimes W)_n$ for all $n \in \mathbb{Z}$ as direct sum and compute its dimension as

$$\dim(V \otimes W)_n = \sum_{i+j=n} \dim(V_i \otimes W_j) = \sum_{i+j=n} \dim V_i \dim W_j = \sum_{i+j=n} a_i b_j,$$

and all elements in $(V \otimes W)_n$ are homogeneous of degree n. Now, the graded dimension of $V \otimes W$ is

$$\operatorname{grdim}(V \otimes W) = \sum_{n\in\mathbb{Z}} \dim(V \otimes W)_n q^n = \sum_{n\in\mathbb{Z}} \sum_{i+j=n} a_i b_j q^n.$$

Thus, we have $\operatorname{grdim}(V \otimes W) = f_1(q)f_2(q)$. □

Proposition 4.9. *Let U, V, W be graded vector spaces, then $U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W)$.*

Proof. The proof is exactly same as Proposition 2.9. □

Lemma 4.10. *Let V, W be vector spaces, then*

$$\wedge^k(V \oplus W) \cong \bigoplus_{i=0}^{k} \wedge^i(V) \otimes \wedge^{k-i}(W).$$

Proof. Let V be an m-dimensional vector space with basis $\{v_1, \dots, v_m\}$ and W be an n-dimensional vector space with basis $\{w_1, \dots, w_n\}$. Then, a basis of $V \oplus W$ is

$$\{e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n}\}$$

where $e_1 = v_1, \dots, e_m = v_m, e_{m+1} = w_1, \dots, e_{m+n} = w_n$. By Theorem 3.5, a basis of $\wedge^k(V \oplus W)$ is $S = \{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} \mid 1 \le i_1 < \dots < i_k \le m+n\}$.

Now, we define a bijective linear transformation

$$T\colon \wedge^k (V \oplus W) \to \bigoplus_{i=0}^{k} \wedge^i(V) \otimes \wedge^{k-i}(W)$$

where $T(e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}) = (e_{i_1} \wedge \dots \wedge e_{i_j}) \otimes (e_{i_{j+1}} \wedge \dots \wedge e_{i_k})$ for each $s = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} \in S$ and $1 \le i_1 < \dots < i_j \le m, m+1 \le i_{i+1} < \dots < i_k \le m+n$ and extend linearly. Then, T is an isomorphism, so the two vector spaces are isomorphic. □

Definition 4.11. (Exterior power of graded vector spaces) Let V be a graded vector space. Then, the k-th *exterior power* of V is a graded vector space with the decomposition

$$\wedge^k(V) = \bigoplus_{n\in\mathbb{Z}} (\wedge^k(V))_n$$

where

$$(\wedge^k(V))_n = \bigoplus_{\substack{k_1 i_1+\cdots+k_m i_m=n,\\ k_1+\cdots+k_m=k}} \wedge^{k_1}(V_{i_1}) \otimes \cdots \otimes \wedge^{k_m}(V_{i_m}).$$

Theorem 4.12. *Let V be a graded vector space with $\operatorname{grdim} V = f(q)a_0 + a_1 q + a_{-1}q^{-1} + a_2 q^2 + a_{-2}q^{-2} + \cdots$ where $a_i = \dim V_i$ for all $i \in \mathbb{Z}$, then the graded dimension of the k-th exterior power of V is*

$$\operatorname{grdim} \wedge^k(V) = \sum_{k_1+\cdots+k_m=k} \binom{a_{i_1}}{k_1} \cdots \binom{a_{i_m}}{k_m} q^{i_1 k_1+\cdots i_m k_m}$$

.

Proof. Let $f(q) = a_0 + a_1 q + a_{-1}q^{-1} + a_2 q^2 + a_{-2}q^{-2} + \cdots$ where $a_i = \dim V_i$ for all $i \in \mathbb{Z}$. Then, we have

$$\begin{aligned}
\dim(\wedge^k(V))_n &= \dim \bigoplus_{\substack{k_1 i_1+\cdots+k_m i_m=n,\\ k_1+\cdots+k_m=k}} \wedge^{k_1}(V_{i_1}) \otimes \cdots \otimes \wedge^{k_m}(V_{i_m}) \\
&= \sum_{\substack{k_1 i_1+\cdots+k_m i_m=n,\\ k_1+\cdots+k_m=k}} \dim(\wedge^{k_1}(V_{i_1}) \otimes \cdots \otimes \wedge^{k_m}(V_{i_m})) \\
&= \sum_{\substack{k_1 i_1+\cdots+k_m i_m=n,\\ k_1+\cdots+k_m=k}} \binom{a_{i_1}}{k_1} \cdots \binom{a_{i_m}}{k_m}
\end{aligned}$$

Therefore,

$$\operatorname{grdim} \wedge^k(V) = \sum_{k_1+\cdots+k_m=k} \binom{a_{i_1}}{k_1} \cdots \binom{a_{i_m}}{k_m} q^{i_1 k_1+\cdots i_m k_m}$$

□

5. λ-Rings

Definition 5.1. (λ-ring) A λ-*ring* is a commutative ring R together with operations $\lambda^n \colon R \to R$ for every non-negative integer n such that

(a) $\lambda^0(r) = 1$ for all $r \in R$
(b) $\lambda^1(r) = r$ for all $r \in R$
(c) $\lambda^n(1) = 0$ for all $n > 1$
(d) $\lambda^n(r+s) = \sum_{k=0}^n \lambda^k(r)\lambda^{n-k}(s)$ for all $r, s \in R$
(e) $\lambda^n(rs) = P_n(\lambda^1(r), \ldots, \lambda^n(r), \lambda^1(s), \ldots, \lambda^n(s))$ for all $r, s \in R$
(f) $\lambda^m(\lambda^n(r)) = P_{m,n}(\lambda^1(r), \ldots, \lambda^{mn}(r))$ for all $r \in R$

where P_n and $P_{m,n}$ are certain universal polynomials with integer coefficients such that if $x = x_1 + x_2 + \cdots$, $y = y_1 + y_2 + \cdots$, then

$$\sum_n P_n(\lambda^1(x), \ldots, \lambda^n(x), \lambda^1(y), \ldots, \lambda^n(y))t^n = \prod_{i,j}(1 + tx_iy_j),$$

i.e. each $P_n(\lambda^1(x), \ldots, \lambda^n(x), \lambda^1(y), \ldots, \lambda^n(y))$ is the coefficient of t^n in the expansion of the right hand side. So, we have $P_0 = 1$, $P_1(\lambda^1(x), \lambda^1(y)) = \sum_{i,j} x_iy_j$, and so on.

$$\sum_m P_{m,n}(\lambda^1(x), \ldots, \lambda^{mn}(x))t^m = \prod_{i_1<i_2<\cdots<i_n}(1 + tx_{i_1}x_{i_2}\cdots x_{i_n}),$$

i.e. each $P_{m,n}(\lambda^1(r), \ldots, \lambda^{mn}(r))$ is the coefficient of t^m in the expansion of the right hand side. So, we have $P_{0,n} = 1$, $P_{1,n}(\lambda^1(x), \ldots, \lambda^n(x)) = \sum_{i_1<i_2<\cdots<i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$, and so on.

Lemma 5.2. *Given any λ-ring R and $x \in R$, $P_{1,0}(x) = 1$ and $P_{m,0}(x) = 0$ for all $m > 1$.*

Proof. Let x be an arbitrary element in R. By axiom (f), $P_{1,0}(x) = \lambda^1(\lambda^0(x))$. Then, by applying axioms (a) and (b), $\lambda^1(\lambda^0(x)) = \lambda^1(1) = 1$. So, $P_{1,0}(x) = 1$. Similarly, let $m > 1$. Then, by axiom (f), (a), and (c), $P_{m,0}(x) = \lambda^m(\lambda^0(x)) = \lambda^m(1) = 0$. □

Example 5.3. The ring $(\mathbb{Z}, +, \cdot)$ together with the operation of binomial coefficients $\lambda^n(x)$ where

$$\lambda^n(x) = \begin{cases} \binom{x}{n} & \text{if } 0 \le x \le n \\ (-1)^n\binom{-x}{n} & \text{if } -n \le x < 0 \\ 0 & \text{elsewhere} \end{cases}$$

is a λ-ring.

The binomial coefficient $\lambda^n(x) = \binom{x}{n}$ satisfies the axioms: (a), (b), (c), and (d) hold by definition and properties of binomial coefficient.

For (e), let $r = r_1 + r_2 + \cdots$ where

$$r_i = \begin{cases} 1 & \text{if } r \ge 0, 1 \le i \le r, \\ -1 & \text{if } r < 0, 1 \le i \le -r, \\ 0 & \text{elsewhere.} \end{cases}$$

Similarly, we define $s = s_1 + s_2 + \cdots$ where

$$s_j = \begin{cases} 1 & \text{if } s \ge 0, 1 \le j \le s, \\ -1 & \text{if } s < 0, 1 \le j \le -s, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, when $r \ge 0, s \ge 0$ or $r < 0, s < 0$,

$$\sum_n P_n(\lambda^1(r), \ldots, \lambda^n(r), \lambda^1(s), \ldots, \lambda^n(s))t^n = \prod_{i,j}(1 + tr_is_j) = (1 + t)^{rs} = \sum_n \binom{rs}{n}t^n.$$

When $r < 0, s \ge 0$ or $r \ge 0, s < 0$,

$$\sum_n P_n(\lambda^1(r), \ldots, \lambda^n(r), \lambda^1(s), \ldots, \lambda^n(s))t^n = \prod_{i,j}(1+tr_is_j) = (1-t)^{-rs} = \sum_n (-1)^n\binom{-rs}{n}t^n.$$

Hence, by equating the coefficients of each t^n from both sides, (e) holds.

For (f), when $r \geq 0$, we have

$$\sum_m P_{m,n}(\lambda^1(r), \ldots, \lambda^{mn}(r))t^m = \prod_{i_1<i_2<\cdots<i_n}(1 + tr_{i_1}r_{i_2}\cdots r_{i_n}) = (1+t)^{\binom{r}{n}} = \sum_m \binom{\binom{r}{n}}{m} t^m.$$

When $r < 0$, we have

$$\sum_m P_{m,n}(\lambda^1(r), \ldots, \lambda^{mn}(r))t^m = \prod_{i_1<i_2<\cdots<i_n}(1+tr_{i_1}r_{i_2}\cdots r_{i_n}) = (1+(-1)^n t)^{\binom{-r}{n}} = \sum_m \binom{(-1)^n\binom{-r}{n}}{m} t^m.$$

Hence, by equating the coefficients of each t^m from both sides, (f) holds.

Thus, all axioms of a λ-ring hold for the binomial coefficient, so $(\mathbb{Z}, +, \cdot)$ with the $\lambda^n(x) = \binom{x}{n}$ is a λ-ring.

Definition 5.4. (Torsion free) In a ring R, an element r is *torsion free* if $nr = 0$ implies $n = 0$.

Definition 5.5. (Binomial ring) A ring R is a *binomial ring* if R is torsion free and the binomial coefficients

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!} \in R$$

for all $r \in R$, and n a positive integer.

Remark 5.6. According to [2, Proposition 8.4], any binomial ring with the operations of binomial coefficients $\lambda^n(x) = \binom{x}{n}$ is a λ-ring. Hence, the ring $(\mathbb{Z}, +, \cdot)$ together with the operation of binomial coefficients is a λ-ring as a special case of binomial rings.

Definition 5.7. (Power series) A *power series* in a ring R, denoted $R[\![t]\!]$, is a series of the form $\sum_{n=0}^{\infty} a_n t^n$ where $a_n \in R$ for all $n \geq 0$ and t is an indeterminate.

Example 5.8. Let R be a ring, then the ring $1 + tR[\![t]\!]$ of power series with constant term 1 where

(a) addition is defined to be multiplication of power series, i.e. $r \oplus s = rs$ for all r, s,
(b) multiplication is defined to be

$$\left(1 + \sum_{n=1}^{\infty} r_n t^n\right) \odot \left(1 + \sum_{n=1}^{\infty} s_n t^n\right) = 1 + \sum_{n=1}^{\infty} P_n(r_1, \ldots, r_n, s_1, \ldots, s_n)t^n,$$

(c) λ-operations are defined to be

$$\lambda^n\left(1 + \sum_{m=1}^{\infty} r_m t^m\right) = 1 + \sum_{m=1}^{\infty} P_{m,n}(r_1, \ldots, r_{mn})t^m$$

is a λ-ring.

The ring $1 + tR[\![t]\!]$ is a ring under the defined addition and multiplication in (a) and (b). The additive identity is 1 and the additive inverse of an arbitrary element $1 + \sum_{n=1}^{\infty} a_n x^n$ is $1 + \sum_{n=1}^{\infty} b_n x^n$, where $b_n = -a_n + \sum_{i=1}^{n-1} a_i b_{n-i}$. Hence, since R is closed under addition and multiplication, $b_1 = -a_1$ and by induction on n, $b_n \in R$, so the additive inverse $1 + \sum_{n=1}^{\infty} b_n x^n \in 1 + tR[\![t]\!]$.

The multiplicative identity of the ring is $1+t$ because for any element $r = 1+\sum_{n=1}^{\infty} r_n t^n$, we have

$$r \odot (1+t) = 1 + \sum_{n=1}^{\infty} P_n(r_1, \ldots, r_n, 1, 0, \ldots, 0)t^n = 1 + \sum_{n=1}^{\infty} r_n t^n = r.$$

By uniqueness of the multiplicative identity, $1+t$ is the multiplicative identity in $1+tR[\![t]\!]$.

The distributive law also holds in this ring because for any element $a = 1 + \sum_{n=1}^{\infty} a_n t^n$, $b = 1 + \sum_{n=1}^{\infty} b_n t^n$, and $c = 1 + \sum_{n=1}^{\infty} c_n t^n$

For axiom (a), we have $\lambda^0(r) = 1+t$, the multiplicative identity, for all $r \in R$.

For axiom (b), we have $\lambda^1(r) = r$ for all $r \in 1+tR[\![t]\!]$, because (need to prove $P_{m,1}(r_1, \ldots, r_m) = r_m$ for all $m > 1$). By replacing n with 1 in (c), $mn = m$ and the right hand side becomes

$$1 + \sum_{m=1}^{\infty} P_{m,1}(r_1, \ldots, r_m)t^m = 1 + \sum_{m=1}^{\infty} r_m t^m = r.$$

For axiom (c), let $n > 1$, since $r_2, r_3, \ldots = 0$, $\sum_{m=1}^{\infty} P_{m,n}(r_1, \ldots, r_{mn})t^m = 0$, so $\lambda^n(1+t) = 1$ which is the additive identity of the ring.

By [1, Theorem 1.4], one can also verify axioms (d), (e), and (f). Thus, $1+tR[\![t]\!]$ is a λ-ring.

Definition 5.9. (Ring of symmetric polynomials) A polynomial $f(x_1, \ldots, x_n)$ in $R[x_1, \ldots, x_n]$ is called a symmetric polynomial if it is unchanged by any permutation of indeterminates, i.e $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$ for all permutations $\sigma \in S_n$.

Definition 5.10. (Ring of symmetric functions) The ring of symmetric functions, denoted Sym, is defined as follows.

1) A degree d symmetric function is a formal (i.e. possibly infinite) linear combination of degree d monomials in $x_1, x_2, \ldots$ that is invariant under permutation. The set of all degree d symmetric functions is denoted Sym_d.
2) Then the ring of symmetric functions has elements which are finite linear combinations of degree d symmetric functions, i.e. $\mathrm{Sym} = \bigoplus_{d=0}^{\infty} \mathrm{Sym}_d$.

Definition 5.11. (Elementary symmetric functions) The elementary symmetric functions in Sym are defined as $\sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$ for any positive integer k.

Remark 5.12. By [3, Theorem 4.5.4], the ring of symmetric functions is isomorphic to the ring of polynomials in the elementary symmetric functions. This is called the fundamental theorem of symmetric functions.

Example 5.13. The ring of symmetric functions together with the operation λ^n is a λ-ring if $\lambda^k(e_1) = e_k$ for all $k \geq 0$ and axiom (d), (e) and (f) in Definition 5.1 hold when r, s are e_r, e_s. Then, λ^k of each symmetric function can be uniquely determined.

By $\lambda^k(e_1) = e_n$ and the definition of $P_{n,k}$, we have

$$\sum_n \lambda^n(e_k)t^n = \prod_{i_1 < \cdots < i_k} (1 + t x_{i_1} \cdots x_{i_k}).$$

From the equation above, we know that $\lambda^0(e_k) = 1$ for all k, so axiom (a) holds. By the fundamental theorem of symmetric functions in Remark 5.12, all symmetric functions are sums of products of elementary symmetric functions. By axiom (e), $\lambda^0(e_r e_s) = 1$ for all r, s. By axiom (f), $\lambda^0(e_r + e_s) = 1$ for all r, s. Thus, $\lambda^0(f) = 1$ for all symmetric functions f. Similarly, we can show that axioms (b) and (c) in Definition 5.1 hold. Similarly, we can apply axiom (d), (e), and (f) to get $\lambda^k(f)$ for any symmetric function f. By [4, Chapter 1], the λ-operation is well defined and axioms (d), (e), (f) hold for all symmetric functions. Thus, the ring of symmetric functions with λ-operation defined above is a λ-ring.

6. Direct Sum, Tensor Product, Exterior Power and λ-Ring Structure

Definition 6.1. (Category) A category C, is an algebraic structure that consists of

(a) a collection of objects, $\mathrm{Ob}(C)$
(b) a class of morphisms, $\mathrm{Hom}_C(a, b)$ for any $a, b \in \mathrm{Ob}(C)$
(c) a composition $\circ\colon \mathrm{Hom}_C(a, b) \circ \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$ for any $a, b, c \in \mathrm{Ob}(C)$

such that the composition is associative and there exists an identity morphism $\mathrm{id}_a \in \mathrm{Hom}_C(a, a)$ for all $a \in \mathrm{Ob}(C)$. That is if $f\colon a \to b$ and $g\colon b \to c$ where $a, b, c \in \mathrm{Ob}(C)$, then $f = \mathrm{id}_b \circ f$ and $g = g \circ \mathrm{id}_b$.

Definition 6.2. (Grothendieck group) Let Vect be the category of finite-dimensional vector spaces, $F(\mathrm{Vect})$ be the free abelian group generated by the isomorphism classes of finite-dimensional vector spaces, and $\langle [U \oplus V] - [U] - [V] \rangle$ be the subgroup generated by $[U \oplus V] - [U] - [V]$ for $U, V \in \mathrm{Ob}(\mathrm{Vect})$, where $[W]$ denotes the isomorphism class of $W \in \mathrm{Ob}(\mathrm{Vect})$. Then, the *Grothendieck group* of Vect is the quotient space $K(\mathrm{Vect}) = F(\mathrm{Vect})/\langle [U \oplus V] = [U] + [V] \rangle$. We will continue to use the symbol $[V]$ for the image of the isomorphism class of V in $K(\mathrm{Vect})$.

Theorem 6.3. *The Grothendieck group of the category* Vect *is isomorphic to the group of integers under addition, i.e.* $K(\mathrm{Vect}) \cong \mathbb{Z}$.

Proof. Define a mapping $f\colon F(\mathrm{Vect}) \to \mathbb{Z}$ by $f(\sum_i a_i[V_i]) = \sum_i a_i \dim(V_i)$ for any vector spaces V_i over F and $a_i \in \mathbb{Z}$. Then, we have $f([U \oplus V] - [U] - [V]) = \dim(U \oplus V) - \dim(U) - \dim(V) = 0$, so $\langle [U \oplus V] = [U] + [V] \rangle \subseteq \ker(f)$.

Let $\sum_{i=1}^n a_i[V_i]$ be an arbitrary element of $\ker(f)$. Then, by relabeling the a_i, we have $\sum_{i=1}^n a_i[V_i] = \sum_{i=1}^k a_i[V_i] - \sum_{i=k+1}^n (-a_i)[V_i]$ where $a_i \geq 0$ for $1 \leq i \leq k$ and $a_i < 0$ for $k+1 \leq i \leq n$. Since $\sum_{i=1}^n a_i[V_i] \in \ker(f)$, $\sum_{i=1}^n a_i \dim(V_i) = \sum_{i=1}^k a_i \dim(V_i) - \sum_{i=k+1}^n (-a_i) \dim(V_i) = 0$. Then, $U = \bigoplus_{i=1}^k V_i^{\oplus a_i}$, $V = \bigoplus_{i=k+1}^n V_i^{\oplus a_i}$, and they have the same dimension, hence are isomorphic. So, in $K(\mathrm{Vect})$, the element of the kernel is $[U] - [V] = 0$, and $\ker(f) \subseteq \langle [U \oplus V] = [U] + [V] \rangle$

Obviously, f is a surjective function and, by the above, $\langle [U \oplus V] - [U] - [V] \rangle = \ker(f)$. Thus, by the first isomorphism theorem, $K(\mathrm{Vect}) \cong \mathbb{Z}$. ∎

$$\begin{tikzcd}
\mathrm{Vect} \arrow[r, "\oplus"] \arrow[d, "K"] & \mathrm{Vect} \arrow[d, "K"] \\
\mathbb{Z} \arrow[r, "+"] & \mathbb{Z}
\end{tikzcd}$$

Example 6.4. An element $3[\mathbb{R}^2] - [\mathbb{R}^5] + 2[\mathbb{R}^7]$ in $K(\text{Vect})$ over the field of real numbers $\mathbb{R}$ equals $15[\mathbb{R}]$ and it maps to 15 in $\mathbb{Z}$ in the isomorphism $K(\text{Vect}) \cong \mathbb{Z}$.

As shown above, the Grothendieck group, $K(\text{Vect})$, is isomorphic to the abelian group $\mathbb{Z}$ under addition. Similarly, we define the multiplication operation on the same $K(\text{Vect})$ to be $[U][V] = [U \otimes V]$. Then, multiplication is well-defined because for any vector spaces U, V, we have for any $[W]$ the product is

$$\begin{aligned}[W]([U \oplus V] - [U] - [V]) &= [W][U \oplus V] - [W][U] - [W][V] \\ &= [W \otimes (U \oplus V)] - [W \otimes U] - [W \otimes U].\end{aligned}$$

By the distributive law of tensor product over direct sum in Proposition 2.9,

$$[W \otimes (U \oplus V)] = [(W \otimes U) \oplus (W \otimes U)] = [W \otimes U] + [W \otimes U].$$

So, we have $[W]([U \oplus V] - [U] - [V]) = 0$. Similarly, $([U \oplus V] - [U] - [V])[W] = 0$. Hence, multiplication is well-defined in $K(\text{Vect})$

Then, we define the λ-operation as $\lambda^k[V] = [\wedge^k(V)]$. It is well-defined because for any vector spaces V,

$$\lambda^k([U \oplus V]) = [\wedge^k(U \oplus V)] = [\bigoplus_{i=0}^{k} \wedge^i(U) \otimes \wedge^{k-i}(V)] = \sum_{i=0}^{k} \lambda^i[U]\lambda[V]$$

. By axiom (d) of λ-ring, that is $\lambda^k([U] + [V])$. Hence, $\lambda^k([U \oplus V] - [U] - [V]) = 0$, and the λ-operation is well-defined.

Hence, the direct sum, tensor product and exterior power on vector spaces induce the structure of a λ-ring of sum, product, and λ-operation in the λ-ring $(\mathbb{Z}, +, \cdot)$ with binomial coefficient as the λ-operation in Example 5.3.

Similarly, we can let grVect be the category of finite-dimensional graded vector spaces discussed in Section 4. Then, let two U, V be two graded vector spaces with graded dimensions $f(q)$ and $g(q)$. The direct sum of U and V has graded dimension $f(q) + g(q)$ which is the sum of two Laurent polynomials from Theorem 4.6. We define the Grothendieck group of the category grVect to be $K(\text{grVect}) = F(\text{grVect})/\langle [U \oplus V] = [U] + [V] \rangle$. Then, we have an analogy of Theorem 6.3.

Theorem 6.5. *The Grothendieck group of the category* grVect *is isomorphic to the group of Laurent polynomials under addition.*

Proof. The proof is similar to the proof of Theorem 6.3 except for the vector spaces are graded vector spaces and $\mathbb{Z}$ is replaced by the ring of Laurent polynomials. $\square$

In this case, the Grothendieck group $K(\text{grVect})$ is isomorphic to Laurent polynomials under addition. In fact, the operations of tensor product and exterior power endow $K(\text{grVect})$ with the structure of a λ-ring. In particular, multiplication is defined as $[U][V] = [U \otimes V]$ for $U, V \in \text{grVect}$. It is well-defined for the same reason as discussed above. The λ-operation is defined to be $\lambda^k[V] = [\wedge^k(V)]$, and it is well-defined for the exact same reason as in the example of vector spaces. Thus, we have the structure of a λ-ring on the ring of Laurent polynomials.

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