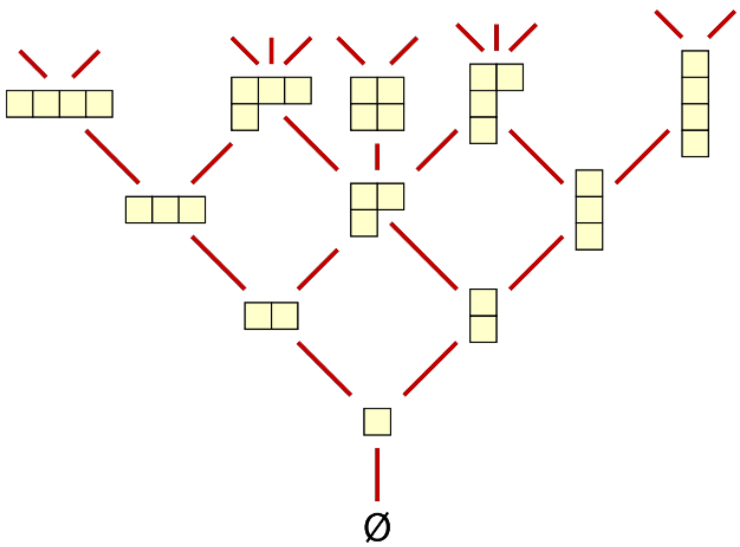
Modern Group Theory

MAT 4199/5145

Fall 2017



Alistair Savage

Department of Mathematics and Statistics

University of Ottawa

Contents

Preface

These are notes for the course *Modern Group Theory* (MAT 4199/5145) at the University of Ottawa. Since the pioneering works of Frobenius, Schur, and Young more than a hundred years ago, the representation theory of the symmetric group has developed into a huge area of study, with applications to algebra, combinatorics, category theory, and mathematical physics. In this course, we will cover the representation theory of the symmetric group following modern techniques developed by Vershik, Olshankii, and Okounkov.

Using techniques from algebra, combinatorics, and category theory, we will cover the following topics.

- *Representation theory of finite groups.* We will begin the course with an introduction to the representation theory of finite groups. This will include a discussion of irreducible representations, tensor products, Schur's lemma, characters, permutation representations, group algebras, and Frobenius reciprocity.
- *The theory of Gelfand-Tsetlin bases.* We will discuss branching rules for representations of symmetric groups and see how such branching rules allow one to obtain particularly nice bases for irreducible representations.
- *The Okounkov-Vershik approach.* We will discuss the combinatorics of Young tableaux, Jucys-Murphy elements, and the Okounkov-Vershik approach to the representation theory of symmetric groups.

Acknowledgements: These notes closely follow the book [CSST10], which is the recommended textbook for the course.

Alistair Savage

Course website: `http://alistairsavage.ca/mat5145`

Chapter 1

Representation theory of finite groups

In this chapter we discuss some basic facts about the representation theory of finite groups. While we will focus on the symmetric groups later in the course, we work mostly with arbitrary finite groups in this chapter. We closely follow the presentation in [CSST10, Ch. 1].

Throughout this chapter, G will denote a finite group and V, W will denote finite-dimensional complex vector spaces. Unless otherwise specified, we will always work over the field of complex numbers. So the term *vector space* means *complex vector space*.

1.1 Basic concepts

In this section we give the main definitions related to representations of finite groups and discuss some examples.

1.1.1 Representations

Recall that the *general linear group*

$$\mathrm{GL}(V) := \{T \colon V \to V : T \text{ is an invertible linear map}\}$$

is a group under composition. Its identity element is the identity map I_V. A (linear) *representation* of G on V is a group homomorphism

$$\sigma \colon G \to \mathrm{GL}(V).$$

The name arises from the fact that elements g of G are "represented" by linear transformations $\sigma(g)$ of V. When we wish to make the vector space V explicit, we will sometimes denote the representation by (σ, V), or simply by V (with the homomorphism σ understood). The *dimension* of the representation σ is the dimension of V.

A subspace $W \leq V$ is said to be *σ-invariant* (or *G-invariant*, when the representation σ is understood) if

$$\sigma(g)W \subseteq W, \quad \text{for all } g \in G \qquad (\text{i.e. } \sigma(g)w \in W \quad \text{for all } g \in G,\ w \in W).$$

If this is the case, then $(\sigma|_W, W)$ is also a representation of G. We say that $\sigma|_W$ is a *subrepresentation* of G. (Note that we will use the notation $\leq$ for subspaces and subgroups. We reserve the symbol $\subseteq$ for set inclusion.)

Note that the trivial spaces V and $\{0\}$ are always invariant. A nonzero representation (σ, V) is *irreducible* if V has no nontrivial invariant subspaces; otherwise we say it is *reducible*.

If (σ, V) is a representation of G and $K \leq G$ is a subgroup, the *restriction* of σ from G to K, denoted $\operatorname{Res}_K^G \sigma$ (or $\operatorname{Res}_K^G V$) is the representation of K of V defined by the restriction $\sigma|_K \colon K \to \mathrm{GL}(V)$.

A *unitary space* is a vector space V endowed with a Hermitian scalar product. Recall that a *Hermitian scalar product* (or *Hermitian inner product*) is a map $\langle \cdot, \cdot \rangle_V \colon V \times V \to \mathbb{C}$ such that, for all $u, v, w \in V$ and $\alpha \in \mathbb{C}$, we have

(a) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,

(b) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$,

(c) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$,

(d) $\langle v, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle$,

(e) $\langle u, v \rangle = \overline{\langle v, u \rangle}$,

(f) $\langle u, u \rangle \geq 0$, with equality only if $u = 0$,

where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. From now on, the term *scalar product* will mean Hermitian scalar product.

Suppose V is a unitary space and $T \colon V \to V$ is a linear operator. The *adjoint* operator T^* is defined by

$$\langle Tu, v \rangle_V = \langle u, T^* v \rangle_V, \quad \text{for all } u, v \in V. \tag{1.1}$$

(See Exercise 1.1.1.)

If V is a unitary space, then a linear operator $T \colon V \to V$ is *unitary* if it preserves the scalar product, i.e. if

$$\langle Tu, Tv \rangle_V = \langle u, v \rangle_V, \quad \text{for all } u, v \in V.$$

(More generally, $T \colon V \to W$ is unitary if $\langle Tu, Tv \rangle_W = \langle u, v \rangle_V$ for all $u, v \in V$.) All unitary operators are invertible (Exercise 1.1.2). Furthermore, $T \in \mathrm{GL}(V)$ is a unitary operator if and only if $T^{-1} = T^*$ (Exercise 1.1.3).

Suppose V is a unitary space. A representation (σ, V) is *unitary* if $\sigma(g)$ is a unitary operator for all $g \in G$ or, in other words, if $\sigma(g^{-1}) = \sigma(g)^*$ for all $g \in G$.

We way a representation (σ, V) is *unitarizable* if there exists a scalar product on V with respect to which σ is unitary.

Lemma 1.1.1. *Every finite-dimensional representation of a finite group is unitarizable.*

Proof. Let $(\cdot, \cdot)$ be an arbitrary scalar product on V. (See Exercise 1.1.4.) Then define a new scalar product on V by

$$\langle u, v \rangle = \sum_{g \in G} (\sigma(g)u, \sigma(g)v), \quad \text{for all } u, v \in V.$$

Then, for all $h \in G$ and $u, v \in V$, we have

$$\begin{aligned} \langle \sigma(h)u, \sigma(h)v \rangle &= \sum_{g \in G} (\sigma(gh)u, \sigma(gh)v) \\ &= \sum_{s \in G} (\sigma(s)u, \sigma(s)v) && \text{(setting } s = gh) \\ &= \langle u, v \rangle. \end{aligned}$$

Hence the representation is unitary with respect to $\langle \cdot, \cdot \rangle$. □

In light of Lemma 1.1.1, we can assume representations are unitary, which we will do from now on. Note that, for infinite groups, it is *not* true that all representations are unitarizable. See for example, this Wikipedia entry.

Exercises.

1.1.1. Prove that, given a linear operator $T\colon V \to V$, there is a *unique* linear operator $T^*\colon V \to V$ satisfying (1.1).

1.1.2. Prove that all unitary operators are invertible.

1.1.3. Prove that $T \in \mathrm{GL}(V)$ is unitary if and only if $T^{-1} = T^*$.

1.1.4. Prove that every finite-dimensional vector space V has a scalar product.

1.1.5. Consider the infinite group $\mathbb{Z}$ (under addition). Let $V = \mathbb{C}^2$, with elements viewed as column vectors. Prove that

$$\sigma\colon \mathbb{Z} \to \mathrm{GL}(V), \quad \sigma(n)(v) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} v$$

is a representation of $\mathbb{Z}$ on V. Prove that this representation is reducible.

1.1.2 Examples

Example 1.1.2 (Trivial representation). The *trivial representation* of G is the one-dimensional representation $(\iota_G, \mathbb{C})$ given by $\iota_G(g) = I_{\mathbb{C}}$, for all $g \in G$.

Example 1.1.3 (Permutation representation (homogeneous space)). Suppose G acts on a finite set X, and let $L(X)$ denote the vector space of all complex-valued functions on X. Then we can define a representation λ of G on $L(X)$ by

$$(\lambda(g)f)(x) = f(g^{-1}x), \quad \text{for all } g \in G,\ f \in L(X),\ x \in X.$$

Note that, for all $g_1, g_2 \in G$, $f \in L(X)$, and $x \in X$, we have

$$\big(\lambda(g_1g_2)(f)\big)(x) = f\big((g_1g_2)^{-1}x\big) = f(g_2^{-1}g_1^{-1}x) = \big(\lambda(g_2)(f)\big)(g^{-1}x) = \Big(\lambda(g_1)\big(\lambda(g_2)f\big)\Big)(x).$$

Hence $\lambda(g_1g_2) = \lambda(g_1)\lambda(g_2)$. Also, it is clear that $\lambda(1_G) = I_{L(X)}$. So λ is indeed a representation. We call it the *permutation representation* of G on $L(X)$.

We can define a scalar product on $L(X)$ by

$$\langle f_1, f_2\rangle = \sum_{x\in X} f_1(x)\overline{f_2(x)}, \quad \text{for all } f_1, f_2 \in L(X). \tag{1.2}$$

With this scalar product, λ is a unitary representation. (See Exercises 1.1.6 and 1.1.7.)

For $x \in X$, the *Dirac function* δ_x centered at x is defined by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Then $\{\delta_X : x \in X\}$ is an orthonormal basis for $L(X)$. In particular,

$$f = \sum_{x\in X} f(x)\delta_x, \quad \text{for all } f \in L(X). \tag{1.3}$$

Furthermore,

$$\lambda(g)\delta_x = \delta_{gx}, \quad \text{for all } g \in G,\ x \in X. \tag{1.4}$$

Example 1.1.4. The group G acts on itself by multiplication on the left:

$$g \cdot h = gh, \quad \text{for all } g, h \in G. \tag{1.5}$$

The associated permutation representation is called the *left regular representation* of G and is typically denoted λ. Explicitly, we have

$$\big(\lambda(g)f\big)(h) = f(g^{-1}h), \quad \text{for all } g, h \in G,\ f \in L(G).$$

Similarly, G acts on itself by multiplication on the right by the inverse:

$$g \cdot h = hg^{-1}, \quad \text{for all } g, h \in G. \tag{1.6}$$

(We must multiply by the inverse in order for this to be a group action. See Exercise 1.1.8.) The associated permutation representation is called the *right regular representation* of G and is typically denoted ρ. Explicitly, we have

$$\big(\rho(g)f\big)(h) = f(hg), \quad \text{for all } g, h \in G,\ f \in L(G).$$

Recall that the *symmetric group* $\mathfrak{S}_n$ of degree n is the group of all bijections (called *permutations*) $\pi\colon \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. Recall that a permutation is *even* (respectively, *odd*) if it is a product of an even (respectively, odd) number of transpositions.

Example 1.1.5 (The sign representation). The *sign representation* (or *alternating representation*) of $\mathfrak{S}_n$ is the one-dimensional representation $(\varepsilon, \mathbb{C})$ defined by

$$\varepsilon(\pi) = \begin{cases} I_{\mathbb{C}} & \text{if } \pi \text{ is even}, \\ -I_{\mathbb{C}} & \text{if } \pi \text{ is odd}. \end{cases}$$

Exercises.

1.1.6. Prove that (1.2) defines a scalar product on $L(X)$.

1.1.7. Prove that the permutation representation λ of Example 1.1.3 is unitary.

1.1.8. Prove that (1.5) and (1.6) define actions of G on the set G.

1.1.3 Intertwining operators

We let $\mathrm{Hom}(V,W)$ denote the vector space of all linear maps from V to W. If (σ,V) and (ρ,W) are two representations of G, we define

$$\mathrm{Hom}_G(V,W)=\mathrm{Hom}_G(\sigma,\rho):=\{T\in\mathrm{Hom}(V,W): T\sigma(g)=\rho(g)T\ \forall\ g\in G\}.$$

We call elements of $\mathrm{Hom}_G(V,W)$ *intertwining operators* (or simply *interwiners*), and we say that they *intertwine* σ and ρ (or V and W).

If σ and ρ are unitary, then

$$\mathrm{Hom}_G(\sigma,\rho)\xrightarrow{\cong}\mathrm{Hom}_G(\rho,\sigma),\quad T\mapsto T^*, \tag{1.7}$$

is an antilinear isomorphism. Here, *antilinear* means that

$$(\alpha T_1+\beta T_2)^*=\bar\alpha T_1^*+\bar\beta T_2^*,\quad \text{for all } \alpha,\beta\in\mathbb{C},\ T_1,T_2\in\mathrm{Hom}_G(\sigma,\rho).$$

To see this, note that $(T^*)^*=T$ (so (1.7) is bijective) and

$$\begin{aligned} T\in\mathrm{Hom}_G(\sigma,\rho) &\iff T\sigma(g)=\rho(g)T\ \forall\ g\in G \\ &\iff \sigma(g)^*T^*=T^*\rho(g)^*\ \forall\ g\in G \qquad \text{(taking the adjoint of both sides)} \\ &\iff \sigma(g^{-1})T^*=T^*\rho(g^{-1})\ \forall\ g\in G \qquad \text{(since } \sigma \text{ and } \rho \text{ are unitary)} \\ &\iff T^*\in\mathrm{Hom}_G(\rho,\sigma). \end{aligned}$$

Two representations (σ,V) and (ρ,W) are *equivalent* if there is a bijective intertwiner $T\in\mathrm{Hom}_G(V,W)$. In this case, we call T an *isomorphism* of representations and we write $\sigma\sim\rho$ or $V\cong W$. If, in addition, σ and ρ are unitary representations and T is a unitary operator, then we say that σ and ρ are *unitarily equivalent*.

Recall that a bijective operator $T\in\mathrm{Hom}(V,W)$ has a unique *polar decomposition* $T=U|T|$, where $|T|\in\mathrm{GL}(V)$ is the (positive definite) square root of the positive operator T^*T, and $U\in\mathrm{Hom}(U,V)$ is unitary. (See, for example, [Roy, Th. 10.5.5] or [Tri, Th. 3.5].)

Lemma 1.1.6. *Two unitary representations of a finite group are equivalent if and only if they are unitarily equivalent.*

Proof. Since unitarily equivalent representations are equivalent by definition, it suffices to prove the reverse implication. Let (σ, V) and (ρ, W) be unitary representations of a finite group G and suppose they are equivalent. Then there exists a bijection $T \in \mathrm{Hom}_G(V, W)$, and

$$T^*T \in \mathrm{Hom}_G(V, V) \cap \mathrm{GL}(V).$$

Let $T = U|T|$ be the polar decomposition of T. Then, for all $g \in G$, we have

$$\begin{aligned}
\big(\sigma(g^{-1})|T|\sigma(g)\big)^2 &= \sigma(g^{-1})|T|\sigma(g)\sigma(g^{-1})|T|\sigma(g) \\
&= \sigma(g^{-1})|T|^2\sigma(g) && \text{(since } \sigma \text{ is a group homomorphism)} \\
&= \sigma(g^{-1})T^*T\sigma(g) && \text{(by the definition of } |T|) \\
&= T^*\rho(g^{-1})\rho(g)T && \text{(since } T \text{ and } T^* \text{ are intertwiners)} \\
&= T^*T && \text{(since } \rho \text{ is a group homomorphism).}
\end{aligned}$$

The uniqueness of positive square root of T^*T implies that

$$\sigma(g^{-1})|T|\sigma(g) = |T| \implies |T|\sigma(g) = \sigma(g)|T|.$$

Hence $|T| \in \mathrm{Hom}_G(V, V)$. Then, for all $g \in G$, we have

$$U\sigma(g) = T|T|^{-1}\sigma(g) = T\sigma(g)|T|^{-1} = \rho(g)U.$$

Thus U is a unitary equivalence from σ to ρ. □

Definition 1.1.7. We let $\mathrm{Irr}(G)$ denote the set of all (unitary) irreducible representations of G, and we let $\widehat{G} = \mathrm{Irr}(G)/\sim$ denote the set of equivalence classes of $\mathrm{Irr}(G)$. By a slight abuse of notation, we will often identity $\widehat{G}$ with a fixed set of representatives of these equivalence classes, that is, a set of irreducible (unitary) pairwise inequivalent representations of G.

Remark 1.1.8. For those students who know a bit of category theory, one can define a category of (finite-dimensional) representations of a fixed group G. The objects are finite-dimensional representations and the morphisms are intertwiners. One can check that the axioms of a category are satisfied. For example, the composition of intertwiners is again an intertwiners, and this composition is associative (see Exercise 1.1.9).

Exercises.

1.1.9. Suppose that (σ_i, V_i), $i = 1, 2, 3$, are representations of G, that $S \in \mathrm{Hom}_G(\sigma_1, \sigma_2)$, and that $T \in \mathrm{Hom}_G(\sigma_2, \sigma_3)$. Prove that the composition TS is an element of $\mathrm{Hom}_G(V_1, V_3)$.

1.1.10. Suppose (σ, V) and (ρ, W) are representations of G. Prove that if $T \in \mathrm{Hom}_G(\sigma, \rho)$ is invertible, then $T^{-1} \in \mathrm{Hom}_G(\rho, \sigma)$.

1.1.4 Direct sums and Maschke's Theorem

Suppose that (σ_j, V_j), $j = 1, 2, \dots, m$, are representations of a group G. Their *direct sum* is the representation $(\sigma, V) = \left(\bigoplus_{j=1}^m \sigma_j, \bigoplus_{j=1}^m V_j\right)$ defined by

$$\sigma(g)(v_j)_{j=1}^m = (\sigma_j(g)v_j)_{j=1}^m .$$

We will often write the element $(v_j)_{j=1}^m$ of a direct sum $\bigoplus_{j=1}^m V_j$ as $\sum_{j=1}^m v_j$.

Conversely, suppose (σ, V) is a representation of G such that

- $V = \bigoplus_{j=1}^m V_j$ (as vector spaces), and
- the subspace V_j is σ-invariant for each $j = 1, \dots, m$.

Then, for $j = 1, \dots, m$, we can define

$$\sigma_j(g) = \sigma(g)|_{V_j} \colon V_j \to V_j, \quad g \in G,$$

and we have

$$\sigma = \bigoplus_{j=1}^m \sigma_j.$$

Theorem 1.1.9 (Maschke's Theorem)**.** *Every finite-dimensional representation of a finite group can be decomposed as a direct sum of irreducible representations. Furthermore, the decomposition can be chosen to be orthogonal (i.e. elements of distinct direct summands are orthogonal to each other).*

Proof. Let (σ, V) be a finite-dimensional representation of G. As noted after Lemma 1.1.1, we may assume that σ is unitary.

We prove the result by induction on the dimension of V. If $\dim V = 0$ or $\dim V = 1$, then V is clearly irreducible. Now suppose $\dim V > 1$, and that the result has been proved for all representations of dimension less than V.

If V is irreducible, we are done. Thus, we suppose V is reducible. Then there is a nontrivial σ-invariant subspace $W \leq V$. We claim that its orthogonal complement

$$W^\perp = \{v \in V : \langle v, w\rangle_V = 0 \ \forall \ w \in W\}$$

is also σ-invariant. Indeed, for $g \in G$ and $v \in W^\perp$, we have

$$\langle \sigma(g)v, w\rangle_V = \langle v, \sigma(g^{-1})w\rangle_V = 0, \quad \text{for all } w \in W,$$

since $\sigma(g^{-1})w \in W$ and $v \in W^\perp$. Thus $\sigma(g)v \in W^\perp$ and so $W^\perp$ is σ-invariant.

By the above, we have the orthogonal decomposition

$$V = W \oplus W^\perp.$$

Since W is nontrivial, we have $\dim W, \dim W^\perp < \dim V$. Therefore, by the inductive hypothesis, W and $W^\perp$ have orthogonal decompositions into direct sums of irreducible representations. This completes the proof of the inductive step. □

Remark 1.1.10. Maschke's Theorem relies on our assumptions about the group and our choice of $\mathbb{C}$ as the ground field. More generally, Maschke's Theorem holds for finite groups and representations over a field whose characteristic does not divide the order of the group. See Exercise 1.1.12.

Remark 1.1.11. A representation that cannot be decomposed into a direct sum of two nontrivial representations is said to be *indecomposable*. Irreducible representations are always indecomposable (for arbitrary groups and arbitrary ground fields). Maschke's Theorem says that, under additional assumptions, the converse is true: indecomposable representations are also irreducible. However, this is not true in general. See Exercise 1.1.11.

Exercises.

1.1.11. Show that the representation of Exercise 1.1.5 cannot be decomposed as a direct sum of irreducible representations. Why does this not violate Maschke's Theorem?

1.1.12. Let p be a prime number. In this exercise, we work over the field $\mathbb{Z}_p$ with p elements, instead of over $\mathbb{C}$. Suppose G is a finite group whose order is divisible by p. Consider the left regular representation $(\lambda, L(G))$ of G. Let

$$V = \left\{ f \in L(G) : \sum_{g \in G} f(g) = 0 \right\}.$$

(a) Prove that V is G-invariant.

(b) Prove that $V \cap W \neq \{0\}$ for every nonzero G-invariant subspace W of $L(G)$. *Hint*: Let f be a nonzero element of W such that $f \notin V$. Define $s \in L(G)$ by $s(g) = 1$ for all $g \in G$. Note that $s \in V$. Prove that $s \in W$ by considering $\sum_{g \in G} \big(\lambda(g) f\big)$.

(c) Explain how this example shows that Maschke's Theorem does not hold when the characteristic of the field divides the order of the group.

1.1.5 The adjoint representation

Recall that the *dual* of the vector space V, denoted V' is the vector space of all linear functions $V \to \mathbb{C}$. If V is endowed with scalar product $\langle \cdot, \cdot \rangle_V$, then we have the *Riesz map*

$$V \to V', \quad v \mapsto \theta_v, \qquad \text{where} \quad \theta_v(w) := \langle w, v \rangle_V, \quad \text{for all } w \in V.$$

This map is antilinear:

$$\theta_{\alpha v + \beta w} = \bar{\alpha}\theta_v + \bar{\beta}\theta_w, \quad \text{for all } \alpha, \beta \in \mathbb{C},\ v, w \in V.$$

Since V is finite-dimensional, this map is also bijective. (This follows from the *Riesz representation theorem*. See, for example, [Tri, Th. 2.1]. Alternatively, you can prove it directly.)

We have dual scalar product on V' defined by

$$\langle \theta_v, \theta_w \rangle_{V'} = \langle w, v \rangle_V, \quad \text{for all } v, w, \in V. \tag{1.8}$$

If $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for V, then the corresponding *dual basis* of V' is $\{\theta_{v_1}, \ldots, \theta_{v_n}\}$. It is characterized by the property that

$$\theta_{v_i}(v_j) = \delta_{i,j}.$$

Suppose (σ, V) is a representation of G. The *adjoint* (or *conjugate*, or *contragredient*) representation is the representation (σ', V') of G defined by

$$(\sigma'(g)f)\, v = f\left(\sigma(g^{-1})v\right), \quad \text{for all } g \in G,\ f \in V', v \in V. \tag{1.9}$$

Note that

$$\begin{aligned}(\sigma'(g)\theta_w)\, v &= \theta_w\left(\sigma(g^{-1})v\right)\\ &= \langle \sigma(g^{-1})v, w\rangle_V\\ &= \langle v, \sigma(g)w\rangle_V\\ &= \theta_{\sigma(g)w}(v).\end{aligned}$$

Thus

$$\sigma'(g)\theta_w = \theta_{\sigma(g)w}, \quad \text{for all } g \in G,\ w \in V. \tag{1.10}$$

It follows that σ is irreducible if and only if σ' is irreducible (Exercise 1.1.13).

Exercises.

1.1.13. Prove that a representation (σ, V) of G is irreducible if and only if the adjoint representation (σ', V') is irreducible.

1.1.6 Matrix coefficients

As we know from linear algebra, if we choose a basis for V, then any linear transformation of V can be represented by a matrix. Let (σ, V) be a unitary representation of G, and let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of V. Then the *matrix coefficients* associated with this basis are given by

$$U_{i,j}(g) = \langle \sigma(g)v_j, v_i\rangle_V, \quad g \in G,\ i, j = 1, \ldots, n.$$

For $g \in G$, let $U(g) = \left(U_{i,j}(g)\right)_{i,j=1}^n \in M_{n,n}(\mathbb{C})$ be the matrix whose entries are the matrix coefficients of (σ, V). Recall that if $M \in M_{n,n}(\mathbb{C})$, then its *conjugate transpose* is the matrix M^* with entries given by

$$M^*_{i,j} = \overline{M_{j,i}}.$$

The matrix M is *unitary* if it is invertible and $M^{-1} = M^*$.

Lemma 1.1.12. *For all $g, g_1, g_2 \in G$, we have*

(a) $U(g_1 g_2) = U(g_1)U(g_2)$,
(b) $U(g^{-1}) = U(g)^*$,
(c) $U(g)$ *is unitary,*
(d) the matrix coefficients of the adjoint representation σ' with respect to the dual basis $\theta_{v_1}, \dots, \theta_{v_n}$ are

$$\langle \sigma'(g)\theta_{v_j}, \theta_{v_i}\rangle_{V'} = \overline{U_{i,j}(g)}.$$

Proof. (a) We have

$$\begin{aligned}
U_{i,j}(g_1 g_2) &= \langle \sigma(g_1 g_2)v_j, v_i\rangle_V \\
&= \langle \sigma(g_1)\sigma(g_2)v_j, v_i\rangle_V && (\sigma \text{ is a group homomorphism}) \\
&= \left\langle \sigma(g_1)\left(\sum_{k=1}^n \langle \sigma(g_2)v_j, v_k\rangle_V v_k\right), v_i \right\rangle_V && \left(w = \sum_{k=1}^n \langle w, v_k\rangle v_k \text{ for all } w \in V\right) \\
&= \sum_{k=1}^n \langle \sigma(g_1)v_k, v_i\rangle_V \langle \sigma(g_2)v_j, v_k\rangle_V && (\text{the scalar product is linear}) \\
&= \sum_{k=1}^n U_{i,k}(g_1)U_{k,j}(g_2).
\end{aligned}$$

(b) We have

$$\begin{aligned}
U_{i,j}(g^{-1}) &= \langle \sigma(g^{-1})v_j, v_i\rangle_V \\
&= \langle v_j, \sigma(g)v_i\rangle_V && (\sigma(g^{-1}) = \sigma(g)^*) \\
&= \overline{\langle \sigma(g)v_i, v_j\rangle_V} && (\text{property of the scalar product}) \\
&= \overline{U_{j,i}(g)}.
\end{aligned}$$

(c) It follows from part (a) that $U(g^{-1}) = U(g)^{-1}$. Then it follows from part (b) that $U(g)$ is unitary.

(d) We have

$$\begin{aligned}
\langle \sigma'(g)\theta_{v_j}, \theta_{v_i}\rangle_{V'} &\overset{(1.10)}{=} \langle \theta_{\sigma(g)v_j}, \theta_{v_i}\rangle_V \\
&\overset{(1.8)}{=} \langle v_i, \sigma(g)v_j\rangle_V \\
&= \overline{\langle \sigma(g)v_j, v_i\rangle_V} && (\text{property of the scalar product}) \\
&= \overline{U_{i,j}(g)}
\end{aligned}$$

□

It follows from Lemma 1.1.12 that we have a group homomorphism

$$U \colon G \to \mathrm{U}(n), \quad g \mapsto U(g),$$

where $\mathrm{U}(n)$ is the group of $n \times n$ unitary matrices (under multiplication). This group homomorphism is called a *unitary matrix realization of σ*. Note that it depends on the choice of orthonormal basis.

Exercises.

1.1.14. Suppose $B = \{v_1, \dots, v_n\}$ and $B' = \{w_1, \dots, w_n\}$ are two orthonormal bases for V. Then we have a "change of basis matrix" A whose (i, j) entry is given by

$$A_{i,j} = \langle v_j, w_i \rangle_V.$$

Let U and U' be the unitary matrix realizations of a representation σ on V in terms of the bases B and B', respectively. State and prove an equality relating $U(g)$, $U'(g)$, and A that holds for all $g \in G$.

1.1.7 Tensor products

In this section, we introduce the important notion of a tensor product. Rather than give the most general possible definition (that of a tensor product of bimodules over rings), we give a more direct definition that is suitable for our purposes.

Suppose V and W are finite-dimensional unitary spaces over $\mathbb{C}$. The *tensor product* $V \otimes W$ is the vector space consisting of all maps

$$B \colon V \times W \to \mathbb{C}$$

that are bi-antilinear:

$$B(\alpha_1 v_1 + \alpha_2 v_2, w) = \bar{\alpha}_1 B(v_1, w) + \bar{\alpha}_2 B(v_2, w),$$
$$B(v, \alpha_1 w_1 + \alpha_2 w_2) = \bar{\alpha}_1 B(v, w_1) + \bar{\alpha}_2 B(v, w_2),$$

for all $\alpha_1, \alpha_2 \in \mathbb{C}$, $v, v_1, v_2 \in V$, and $w, w_1, w_2 \in W$.

For $v \in V$ and $w \in W$ we define the *simple tensor* $v \otimes w \in V \otimes W$ by

$$(v \otimes w)(v_1, w_1) = \langle v, v_1 \rangle_V \langle w, w_1 \rangle_W.$$

The map

$$V \times W \to V \otimes W, \quad (v, w) \mapsto v \otimes w,$$

is bilinear:

$$(\alpha_1 v_1 + \alpha_2 v_2) \otimes (\beta_1 w_1 + \beta_2 w_2) = \sum_{i,j=1}^{2} \alpha_i \beta_j v_i \otimes w_j,$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$, $v_1, v_2 \in V$, and $w_1, w_2 \in W$.

Lemma 1.1.13. *If $\{v_1, \dots, v_n\}$ is an orthonormal basis for V and $\{w_1, \dots, w_m\}$ is an orthonormal basis for W, then*

$$\{v_i \otimes w_j : 1 \le i \le n,\ 1 \le j \le m\} \tag{1.11}$$

is a basis for $V \otimes W$. In particular,

$$\dim(V \otimes W) = (\dim V)(\dim W).$$

Proof. Suppose $B \in V \otimes W$, $v \in V$, and $w \in W$. Then there exist $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \mathbb{C}$ such that

$$v = \sum_{i=1}^{n} \alpha_i v_i \quad \text{and} \quad w = \sum_{i=1}^{m} \beta_i w_i.$$

Then, for $i = 1, \dots, n$, we have

$$\langle v_i, v \rangle_V = \sum_{k=1}^{n} \bar{\alpha}_k \langle v_i, v_k \rangle_V = \bar{\alpha}_k.$$

Similarly, for $j = 1, \dots, m$, we have $\langle w_j, w \rangle_W = \bar{\beta}_j$. Therefore,

$$B(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{\alpha}_i \bar{\beta}_j B(v_i, w_j) = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} B(v_i, w_j) v_i \otimes w_j \right)(v, w).$$

It follows that $B = \sum_{i=1}^{n} \sum_{j=1}^{m} B(v_i, w_j) v_i \otimes w_j$. So every element of $V \otimes W$ can be written in a unique way as a linear combination of the elements of (1.11). Thus, these elements form a basis for $V \otimes W$. □

Remark 1.1.14. It is important to note that not every element of $V \otimes W$ can be written as a simple tensor. When working with an arbitrary element of a tensor product, one must consider finite sums of simple tensors.

In the notation of Lemma 1.1.13, we define a scalar product on $V \otimes W$ by

$$\langle v_i \otimes w_k, v_j \otimes w_\ell \rangle = \langle v_i, v_j \rangle_V \cdot \langle w_k, w_\ell \rangle_W = \delta_{i,j} \delta_{k,\ell}, \tag{1.12}$$

and extending by linearity. Thus the basis (1.11) is orthonormal.

Suppose V_1, V_2, W_1, W_2 are unitary spaces. If $T \in \operatorname{Hom}(V_1, V_2)$ and $S \in \operatorname{Hom}(W_1, W_2)$, we define

$$T \otimes S \in \operatorname{Hom}(V_1 \otimes W_1, V_2 \otimes W_2), \quad (T \otimes S)(v \otimes w) = (Tv) \otimes (Sw), \quad v \in V_1,\ w \in W_1,$$

and extend by linearity.

Lemma 1.1.15. *We have an isomorphism of vector spaces*

$$\operatorname{Hom}(V_1 \otimes W_1, V_2 \otimes W_2) \cong \operatorname{Hom}(V_1, V_2) \otimes \operatorname{Hom}(W_1, W_2).$$

Proof. Since the dimensions of the two spaces are both $(\dim V_1)(\dim V_2)(\dim W_1)(\dim W_2)$, it suffices to prove that every element of $\operatorname{Hom}(V_1 \otimes W_1, V_2 \otimes W_2)$ is a linear combination of elements of the form $T \otimes S$ for $T \in \operatorname{Hom}(V_1, V_2)$ and $S \in \operatorname{Hom}(W_1, W_2)$. For $i = 1, 2$, let $\{v_{i,1}, \dots, v_{i,n_i}\}$ be a basis of V_i, and let $\{w_{i,1}, \dots, w_{i,m_i}\}$ be a basis of W_i. Choose

$$a \in \{1, \dots, n_1\},\ b \in \{1, \dots, n_2\},\ c \in \{1, \dots, m_1\},\ d \in \{1, \dots, m_2\}.$$

Define

$$T_{a,c} \in \operatorname{Hom}(V_1, W_1), \quad T(v_{1,i}) = \begin{cases} w_{1,c} & \text{if } i = a, \\ 0 & \text{if } i \neq a, \end{cases}$$

$$S_{b,d} \in \mathrm{Hom}(V_2, W_2), \quad T(v_{2,j}) = \begin{cases} w_{2,d} & \text{if } j = b, \\ 0 & \text{if } j \neq b, \end{cases}$$

Then

$$(T_{a,c} \otimes S_{b,d})(v_{1,i} \otimes v_{2,j}) = \begin{cases} w_{1,c} \otimes w_{2,d} & \text{if } i = a, \ j = b, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{v_{1,i} \otimes v_{2,j} : 1 \leq i \leq n_1, \ 1 \leq j \leq n_2\}$ is a basis of $V_1 \otimes V_2$ and $\{w_{1,i} \otimes w_{2,j} : 1 \leq i \leq m_1, \ 1 \leq j \leq m_2\}$ is a basis of $W_1 \otimes W_2$, very element of $\mathrm{Hom}(V_1 \otimes V_2, W_1 \otimes W_2)$ can be written as a linear combination of the above maps (for various choices of a, b, c, d). This completes the proof. □

Now suppose G_1 and G_2 are finite groups. Let (σ_i, V_i) be a representation of G_i for $i = 1, 2$. The *outer tensor product* of σ_1 and σ_2 is the representation $\sigma_1 \boxtimes \sigma_2$ of $G_1 \times G_2$ given by

$$\sigma_1 \boxtimes \sigma_2 \colon G_1 \times G_2 \to \mathrm{GL}(V_1 \otimes V_2),$$
$$(\sigma_1 \boxtimes \sigma_2)(g_1, g_2) = \sigma_1(g_1) \otimes \sigma_2(g_2), \quad \text{for all } g_1 \in G_1, \ g_2 \in G_2.$$

Recall that we have the diagonal embedding (a group homomorphism)

$$G \to G \times G, \quad g \mapsto (g, g).$$

Suppose (σ_1, V_1) and (σ_2, V_2) are two representations of the *same* group G. Then the *internal tensor product* of σ_1 and σ_2, denoted $\sigma_1 \otimes \sigma_2$, is the representation of G given by the composition

$$G \to G \times G \xrightarrow{\sigma_1 \boxtimes \sigma_2} \mathrm{GL}(V_1 \otimes V_2).$$

In other words, the $\sigma_1 \otimes \sigma_2$ is given by

$$(\sigma_1 \otimes \sigma_2)(g) = \sigma_1(g) \otimes \sigma_2(g).$$

Exercises.

1.1.15. Suppose (σ_i, V_i) is a representation of G_i for $i = 1, 2$. Let W_i be a σ_i-invariant subspace of V_i for $i = 1, 2$. Prove that $W_1 \otimes W_2$ is a $\sigma_1 \boxtimes \sigma_2$-invariant subspace of $V_1 \otimes V_2$. (Here we identify $W_1 \otimes W_2$ with the subspace of $V_1 \otimes V_2$ spanned by $\{w_1 \otimes w_2 : w_1 \in W_1, \ w_2 \in W_2\}$.)

1.1.16. Prove that the scalar product defined in (1.12) is indeed a Hermitian scalar product on $V \otimes W$.

1.1.8 Cyclic and invariant vectors

Let (σ, V) be a unitary representation of G. For $v \in V$,

$$\langle \sigma(g)v : g \in G \rangle$$

is a σ-invariant subspace of V, called the subspace *generated* by v. (We use the notation $\langle\ \rangle$ to denote the $\mathbb{C}$-span.) If this space is *all* of V, then we say that v is a *cyclic* vector.

We say that a vector $v \in V$ is *σ-invariant* or *fixed* if

$$\sigma(g)v = v, \quad \text{for all } g \in G.$$

We let

$$V^G = \{v \in V : \sigma(g)v = v\ \forall\ g \in G\}$$

denote the subspace of all σ-invariant vectors. More generally, if $K \leq G$ is a subgroup, we let

$$V^K := \{v \in V : \sigma(k)v = v\ \forall\ k \in K\} \tag{1.13}$$

be the subspace of *K-invariant vectors*.

Lemma 1.1.16. *Suppose that $u \in V^G$, $u \neq 0$. If $v \in V$ is orthogonal to u (i.e. $\langle u, v\rangle_V = 0$), then v is not cyclic.*

Proof. For all $g \in G$, we have

$$\begin{aligned}
\langle u, \sigma(g)v\rangle_V &= \langle \sigma(g^{-1})u, v\rangle_V \\
&= \langle u, v\rangle_V && (\text{since } u \in V^G) \\
&= 0.
\end{aligned}$$

Hence $\sigma(g)v \in \langle u\rangle^\perp$. So

$$\langle \sigma(g)v : g \in G\rangle \subseteq \langle u\rangle^\perp \subsetneq V,$$

and hence v is not cyclic. □

The following corollary will be useful in our study of the representation theory of the symmetric group.

Corollary 1.1.17. *Suppose that there exists a cyclic vector $v \in V$, and $g \in G$, $\lambda \in \mathbb{C}$, $\lambda \neq 1$, such that $\sigma(g)v = \lambda v$. Then $V^G = \{0\}$.*

Proof. Suppose $u \in V^G$. Thus $\sigma(g)u = u$. So u and v are eigenvectors for the unitary operator with distinct eigenvalues. Hence there are orthogonal. By Lemma 1.1.16, $u = 0$. □

Exercises.

1.1.17. Show that if $\dim V^G \geq 2$, then V has no cyclic vectors. *Hint*: Suppose $u, w \in V^G$ are nonzero and orthogonal. Then, for $v \in V \setminus V^G$, we have $\dim\langle u, w, v\rangle = 3$, and so there exists a nonzero $u_0 \in \langle u, w\rangle \subseteq V^G$ such that $\langle u_0, v\rangle = 0$.

1.1.18. Suppose G acts transitively on a set X.

(a) Show that $\dim L(X)^G = 1$.

(b) Show that the vectors δ_x are cyclic.

1.2 Schur's lemma and the commutant

In this section we prove Schur's lemma, which is a very useful result in representation theory, even though its proof is quite simple. We also discuss decompositions of representations into isotypic components, commutants, and how these relate to spaces of intertwiners.

1.2.1 Schur's lemma

Lemma 1.2.1 (Schur's lemma)**.** *Suppose (σ, V) and (ρ, W) are irreducible representations of G.*

(a) Every nonzero element of $\operatorname{Hom}_G(\sigma, \rho)$ is an isomorphism.

(b) We have $\operatorname{Hom}_G(\sigma, \sigma) = \mathbb{C} I_V$.

Proof. (a) Suppose $T \in \operatorname{Hom}_G(\sigma, \rho)$. Then $\operatorname{Ker} T \leq V$ and $\operatorname{Im} T \leq W$ are G-invariant (Exercise 1.2.1). Since V is irreducible, this implies that $\operatorname{Ker} T = V$ or $\operatorname{Ker} T = 0$. If $\operatorname{Ker} T = V$, then $T = 0$. Otherwise, T is injective. Also, if $T \neq 0$, then $\operatorname{Im} T \neq 0$, and so $\operatorname{Im} T = W$ since W is irreducible. So T is also surjective.

(b) Suppose $T \in \operatorname{Hom}_G(\sigma, \sigma)$. Since $\mathbb{C}$ is algebraically closed, T has at least one eigenvalue. So there exists $\lambda \in \mathbb{C}$ such that $\operatorname{Ker}(T - \lambda I_V) \neq 0$. This implies that $T - \lambda I_V$ is not injective. So, by part (a), $T - \lambda I_V = 0$. Hence $T = \lambda I_V$.

□

Corollary 1.2.2. *If σ and ρ are irreducible representations of G, then*

$$\dim \operatorname{Hom}_G(\sigma, \rho) = \begin{cases} 1 & \text{if } \sigma \sim \rho, \\ 0 & \text{if } \sigma \not\sim \rho. \end{cases}$$

Corollary 1.2.3. *Every irreducible representation of an abelian group is one-dimensional.*

Proof. Suppose (σ, V) is an irreducible representation of an abelian group G. For every $h \in G$, we have

$$\sigma(h)\sigma(g) = \sigma(hg) = \sigma(gh) = \sigma(g)\sigma(h), \quad \text{for all } g \in G.$$

Thus $\sigma(h) \in \mathrm{Hom}_G(\sigma, \sigma)$. By Lemma 1.2.1(b), we have $\sigma(h) = \chi(h) I_V$ for some $\chi(h) \in \mathbb{C}$. Since this holds for all $h \in G$, it follows that every subspace of V is invariant. Since V is irreducible, we must have $\dim V = 1$. □

Example 1.2.4 (Representations of cyclic groups). Suppose

$$C_n = \langle a : a^n = 1 \rangle$$

is a cyclic group of order n. By Corollary 1.2.3, every irreducible representation corresponds to a group homomorphism

$$\chi \colon C_n \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times := \mathbb{C} \setminus \{0\}.$$

Such a map is uniquely determined by $\chi(a)$. Since the only relation in the group is $a^n = 1$, we can choose any value for $\chi(a)$ such that $\chi(a)^n = 1$, that is, such that $\chi(a)$ is an n-th root of unity. So every representation is of the form

$$\chi_j \colon C_n \to \mathbb{C}, \quad \chi_j\left(a^k\right) = \chi_j(a)^k = \exp\left(k\frac{2\pi i j}{n}\right), \quad k \in \mathbb{Z},$$

for some $j \in \{0, 1, \dots, n-1\}$. Since one-dimensional representations are equivalent if and only if they corresponds to the same χ (Exercise 1.2.2), we have

$$\widehat{C_n} = \{\chi_0, \dots, \chi_{n-1}\}.$$

In particular, the only irreducible representations of C_2, up to isomorphism are the trivial representation $\chi_0 = \iota_{C_2}$ (Example 1.1.2) and the sign representation $\chi_1 = \varepsilon$ (Example 1.1.5).

Exercises.

1.2.1. Suppose (σ, V) and (ρ, W) are representations of G and $T \in \mathrm{Hom}_G(V, W)$. Prove that $\operatorname{Ker} T$ is a σ-invariant subspace of V and that $\operatorname{Im} T$ is a ρ-invariant subspace of W.

1.2.2. Prove that two one-dimensional representations $(\sigma, \mathbb{C})$ and $(\rho, \mathbb{C})$ are equivalent if and only if they are equal.

1.2.2 Multiplicities and isotypic components

A linear transformation $E \in \mathrm{Hom}(V, V)$ is a *projection* if it is *idempotent*: $E^2 = E$. If, in addition, $\mathrm{Im}\, E$ is orthogonal to $\mathrm{Ker}\, E$, we say that E is an *orthogonal projection* of V onto $\mathrm{Im}\, E$. It is not hard to verify that a projection E is orthogonal if and only if it is *self-adjoint*, that is, $E = E^*$. (See Exercise 1.2.3.)

Now suppose (σ, V) is a representation of G. By Maschke's Theorem (Theorem 1.1.9), we have an orthogonal decomposition

$$V = \bigoplus_{\rho \in \widehat{G}} V_\rho,$$

where

$$V_\rho \cong W_\rho^{\oplus m_\rho} := \underbrace{W_\rho \oplus \cdots \oplus W_\rho}_{m_\rho \text{ summands}}$$

is an orthogonal direct sum of m_ρ copies of an irreducible representation W_ρ in the equivalence class $\rho \in \widehat{G}$, for some $m_\rho \in \mathbb{N} = \mathbb{Z}_{\geq 0}$. We adopt the convention that $U^{\oplus 0} = 0$ for a vector space U. The summand V_ρ is called the *ρ-isotypic component* of V, and m_ρ is called the *multiplicity* of ρ in σ (or of W_ρ in V).

Let

$$\widehat{G}_\sigma = \{\rho \in \widehat{G} : m_\rho \geq 1\}$$

be the set of isomorphism classes of irreducible representations of G that appear with nonzero multiplicity in σ. The inclusions of the summands yield interwiners

$$I_{\rho,j} \in \mathrm{Hom}_G(W_\rho, V), \quad \rho \in \widehat{G}_\sigma,\ 1 \leq j \leq m_\rho.$$

So we have

$$V = \bigoplus_{\rho \in \widehat{G}_\sigma} \bigoplus_{j=1}^{m_\rho} I_{\rho,j} W_\rho. \tag{1.14}$$

Thus, every $v \in V$ can be written uniquely in the form

$$v = \sum_{\rho \in \widehat{G}_\sigma} \sum_{j=1}^{m_\rho} v_{\rho,j}, \quad v_{\rho,j} \in I_{\rho,j} W_\rho.$$

For each $\rho \in \widehat{G}_\sigma$ and $1 \leq j \leq m_\rho$, we have the orthogonal projection

$$E_{\rho,j} \in \mathrm{Hom}_G(V, V), \quad E_{\rho,j}\left(\sum_{\eta \in \widehat{G}_\sigma} \sum_{j=1}^{m_\eta} v_{\eta,j}\right) = v_{\rho,j}.$$

It follows that

$$I_V = \sum_{\rho \in \widehat{G}_\sigma} \sum_{j=1}^{m_\rho} E_{\rho,j}. \tag{1.15}$$

Lemma 1.2.5. *For $\rho \in \widehat{G}_\sigma$, the intertwiners $I_{\rho,1}, \dots, I_{\rho,m_\rho}$ form a basis of $\mathrm{Hom}_G(W_\rho, V)$. In particular, $m_\rho = \dim \mathrm{Hom}_G(W_\rho, V)$.*

Proof. Suppose $T \in \mathrm{Hom}_G(W_\rho, V)$. Then

$$T = I_V T = \sum_{\eta \in \widehat{G}_\sigma} \sum_{j=1}^{m_\eta} E_{\eta,j} T.$$

The domain of $E_{\eta,j}T$ is W_ρ, while its image is $I_{\eta,j}W_\eta \cong W_\eta$. Thus, by Schur's lemma (Lemma 1.2.1), we have

$$E_{\eta,j}T = \begin{cases} 0, & \text{if } \eta \neq \rho, \\ \alpha_j I_{\rho,j} \text{ for some } \alpha_j \in \mathbb{C}, & \text{if } \eta = \rho. \end{cases}$$

Hence,

$$T = \sum_{j=1}^{m_\rho} \alpha_j I_{\rho,j}.$$

Since this decomposition is unique, the lemma follows. □

Corollary 1.2.6. *With notation as in Lemma 1.2.5, we have $m_\rho = \dim \mathrm{Hom}_G(V, W_\rho)$.*

Proof. This follows from Lemma 1.2.5 and (1.7). Alternatively, it can be proved directly, using an argument analogous to that used to prove Lemma 1.2.5 (Exercise 1.2.4). □

The isotypic summands V_ρ of V are unique. However, the decomposition of V_ρ into a sum of irreducible representations is not unique when $m_\rho > 1$; it corresponds to a choice of basis for $\mathrm{Hom}_G(W_\rho, V)$.

For $\rho \in \widehat{G}_\sigma$ and $1 \leq j, k \leq m_\rho$, consider the intertwiner $T^\rho_{k,j} \in \mathrm{Hom}_G(V, V)$ defined to be the composition

$$T^\rho_{k,j} \colon V \twoheadrightarrow I_{\rho,j}W_\rho \xrightarrow{I_{\rho,k}I^{-1}_{\rho,j}} I_{\rho,k}W_\rho \hookrightarrow V, \tag{1.16}$$

where $\twoheadrightarrow$ and $\hookrightarrow$ denote the projection onto and inclusion of the given summand of V, and $I^{-1}_{\rho,j}$ denotes the inverse of $I_{\rho,j}$ when its codomain is replaced by its image (so that it becomes invertible, since it is injective). It follows that

$$T^\rho_{k,j} T^\eta_{s,t} = \delta_{\rho,\eta}\delta_{j,s} T^\rho_{k,t}. \tag{1.17}$$

Note also that $T^\rho_{j,j} = E_{\rho,j}$. Furthermore, it follows from Corollary 1.2.2 that

$$\mathrm{Hom}_G(I_{\rho,j}W_\rho, I_{\rho,k}W_\rho) = \mathbb{C}T^\rho_{k,j}.$$

Exercises.

1.2.3. Prove that a projection is orthogonal if and only if it is self-adjoint.

1.2.4. Prove Corollary 1.2.6 directly, using an argument analogous to that used to prove Lemma 1.2.5.

1.2.5. Prove that a representation (σ, V) is reducible if and only if $\mathrm{Hom}_G(\sigma, \sigma)$ has nontrivial idempotents. (The trivial idempotents are 0 and I_V.)

1.2.3 Finite-dimensional algebras

An *(associative) algebra* over $\mathbb{C}$ is a vector space $\mathcal{A}$ with bilinear product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $\mathcal{A}$ is a ring (possibly without unit) with respect to this product and the vector addition.

A *subalgebra* of $\mathcal{A}$ is a subspace $\mathcal{B} \leq \mathcal{A}$ that is closed under multiplication. An *involution* of $\mathcal{A}$ is a bijective map $A \mapsto A^*$ such that

- $(A^*)^* = A$,
- $(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$, and
- $(AB)^* = B^* A^*$,

for all $\alpha, \beta \in \mathbb{C}$ and $A, B \in \mathcal{A}$. An algebra with involution is called an *involutive algebra* or $*$-*algebra*. An element A in an involutive algebra is *self-adjoint* if $A^* = A$.

The algebra $\mathcal{A}$ is *commutative* if it is commutative as a ring:

$$AB = BA, \quad \text{for all } A, B \in \mathcal{A}.$$

The *center* of $\mathcal{A}$ is the commutative algebra

$$Z(\mathcal{A}) = \{B \in \mathcal{A} : AB = BA \quad \text{for all } A \in \mathcal{A}\}.$$

The algebra $\mathcal{A}$ is *unital* if there exists $I \in \mathcal{A}$ such that

$$AI = IA = A, \quad \text{for all } A \in \mathcal{A}.$$

The unit is unique: If I' is another unit, then

$$I = II' = I'.$$

Furthermore,

$$I^* = I^* I = ((I^* I)^*)^* = (I^*(I^*)^*)^* = (I^* I)^* = (I^*)^* = I.$$

So the unit is self-adjoint.

Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebras. A map $\phi \colon \mathcal{A}_1 \to \mathcal{A}_2$ is an *algebra homomorphism* if

- ϕ is linear,

- ϕ is multiplicative: $\phi(AB) = \phi(A)\phi(B)$ for all $A, B \in \mathcal{A}_1$.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are involutive, then ϕ is a $*$-*homomorphism* if it is an algebra homomorphism and it preserves the involution:

$$\phi(A^*) = \phi(A)^*, \quad \text{for all } A \in \mathcal{A}_1.$$

If, in addition, ϕ is bijective, then we call it a $*$-*isomorphism*, and we say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are $*$-*isomorphic*.

A map $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ is a $*$-*anti-homomorphism* if it satisfies the conditions of a $*$-homomorphism, except that we replace the multiplicative property by the *anti-multiplicative property*

$$\phi(AB) = \phi(B)\phi(A).$$

If ϕ is also bijective, it is called a $*$-*anti-isomorphism*, and we say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are $*$-*anti-isomorphic*..

Suppose $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$, $k \geq 2$, are algebras. Their *direct sum*, denoted $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_k$, is equal to $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_k$ as a vector space, with componentwise product:

$$(A_1, \ldots, A_k)(B_1, \ldots, B_k) = (A_1B_1, \ldots, A_kB_k), \quad \text{for all } A_i, B_i \in \mathcal{A}_i,\ 1 \leq i \leq k.$$

The algebra *generated* by a subset $B \subseteq \mathcal{A}$, denoted $\langle B \rangle$, is the smallest subalgebra of $\mathcal{A}$ containing B. Explicitly, $\langle B \rangle$ is the set of all linear combinations of products of elements of B.

The dimension of $\mathcal{A}$ is its dimension as a vector space. Suppose $\mathcal{A}$ is finite dimensional, and let $\{e_1, \ldots, e_d\}$ be a basis of $\mathcal{A}$. Then the *structure coefficients* $c_{i,j,k} \in \mathbb{C}$, $1 \leq i, j, k \leq d$ defined by

$$e_ie_j = \sum_{k=1}^{d} c_{i,j,k}e_k$$

uniquely determine the product in $\mathcal{A}$.

Example 1.2.7 (Endomorphism algebra). The *endomorphism algebra* $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ is a unital $*$-algebra with the usual vector space structure and multiplication (composition of operators). The involution is the map $T \mapsto T^*$, where T^* is the adjoint of T. Moreover, if (σ, V) is a unitary representation of G, then the subalgebra $\mathrm{Hom}_G(V, V)$ of $\mathrm{End}(V)$ is also a unital $*$-algebra.

Example 1.2.8 (Matrix algebra). The *matrix algebra* $M_{m,m}(\mathbb{C})$ of complex $m \times m$ matrices is a unital $*$-algebra under matrix multiplication, where the involution is givin by the conjugate transpose. If $V = \mathbb{C}^m$, then $M_{m,m}(\mathbb{C}) \cong \mathrm{End}(V)$ ($*$-isomorphism). The center of $M_{m,m}(\mathbb{C})$ is

$$Z(M_{m,m}(\mathbb{C})) = \{\lambda I : \lambda \in \mathbb{C}\} \cong \mathbb{C},$$

where I is the identity matrix (Exercise 1.2.6).

Exercises.

1.2.6. Prove that $Z(M_{m,m}(\mathbb{C})) = \{\lambda I : \lambda \in \mathbb{C}\}$.

1.2.7. Prove that if $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ is a homomorphism of algebras, then $\operatorname{Im}\phi$ is a subalgebra of $\mathcal{A}_2$. If, in addition, ϕ is a $*$-homomorphism, prove that $\operatorname{Im}\phi$ is an involutive algebra, with involution induced by the involution on $\mathcal{A}_2$.

1.2.8. A subspace J of an algebra $\mathcal{A}$ is a *(two-sided) ideal* of $\mathcal{A}$ if

$$AB \in J \text{ and } BA \in J, \quad \text{for all } A \in \mathcal{A}, B \in J.$$

(a) Prove that if $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ is a homomorphism of algebras, then $\operatorname{Ker}\phi$ is an ideal of $\mathcal{A}_1$.

(b) If J is an ideal of $\mathcal{A}$, define a natural algebra structure on the quotient vector space $\mathcal{A}/J$.

(c) If J is an ideal of $\mathcal{A}$ that is $*$-invariant (i.e. $A^* \in J$ for all $A \in J$), define a natural involution on the quotient algebra $\mathcal{A}/J$ and prove that this gives the quotient the structure of an involutive algebra.

(d) If $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ is a $*$-homomorphism, prove that $\operatorname{Im}\phi \cong \mathcal{A}_1/\operatorname{Ker}\phi$ as $*$-algebras.

1.2.9. Let $m \geq 2$ and consider the algebra $\mathbb{C}[x]/(x^m)$, where $(x^m) = \mathbb{C}[x]x^m$ is the ideal of $\mathbb{C}[x]$ generated by x^m. Prove that $\mathbb{C}[x]/(x^m)$ is *not* isomorphic as an algebra to $\mathbb{C}^m = \bigoplus_{k=1}^m \mathbb{C}$, even though these two algebras are both commutative and both have dimension m. *Hint*: The image of x in the quotient $\mathbb{C}[x]/(x^m)$ has a property that no element of $\mathbb{C}^m$ has.

1.2.10. An element E of an algebra $\mathcal{A}$ is *idempotent* if $E^2 = E$. Of course, every unital algebra has the *trivial* idempotents 0 and I. Suppose a unital algebra $\mathcal{A}$ has a nontrivial central idempotent E, that is, E is a nontrivial idempotent in the center of $\mathcal{A}$. Show that

$$E\mathcal{A}E := \{EAE : A \in \mathcal{A}\} \quad \text{and} \quad (I-E)\mathcal{A}(I-E) := \{(I-E)A(I-E) : A \in \mathcal{A}\}$$

are subalgebras of $\mathcal{A}$ with units. (Here we do *not* require the units of the subalgebras to be the same as the unit of $\mathcal{A}$.) Furthermore, prove that $\mathcal{A} = E\mathcal{A}E \oplus (I-E)\mathcal{A}(I-E)$ (direct sum of algebras). Thus, central idempotents allow one to decompose algebras (or rings) as direct sums of smaller algebras (or rings).

1.2.4 The commutant

Suppose (σ, V) is a representation of G.

Definition 1.2.9 (Commutant)**.** The algebra $\operatorname{End}_G(V) := \operatorname{Hom}_G(V,V)$ is called the *commutant* of (σ, V).

Recall the elements $T^\rho_{k,j} \in \operatorname{End}_G(V)$ defined in (1.16).

Theorem 1.2.10. *The set*

$$\{T^\rho_{k,j} : \rho \in \widehat{G}_\sigma,\ 1 \le k, j \le m_\rho\} \tag{1.18}$$

is a basis for $\operatorname{End}_G(V)$*. Furthermore, the map*

$$\operatorname{End}_G(V) \to \bigoplus_{\rho\in\widehat{G}_\sigma} M_{m_\rho,m_\rho}(\mathbb{C}), \quad \sum_{\rho\in\widehat{G}_\sigma}\sum_{k,j=1}^{m_\rho} \alpha^\rho_{k,j}T^\rho_{k,j} \mapsto \bigoplus_{\rho\in\widehat{G}_\sigma} \left(\alpha^\rho_{k,j}\right)_{k,j=1}^{m_\rho}, \tag{1.19}$$

is an isomorphism of algebras.

Proof. Suppose $T \in \operatorname{End}_G(V)$. Then

$$T = I_V T I_V \overset{(1.15)}{=} \left(\sum_{\rho\in\widehat{G}}\sum_{k=1}^{m_\rho} E_{\rho,k}\right) T \left(\sum_{\eta\in\widehat{G}}\sum_{j=1}^{m_\eta} E_{\eta,j}\right) = \sum_{\rho,\eta\in\widehat{G}}\sum_{k=1}^{m_\rho}\sum_{j=1}^{m_\eta} E_{\rho,k}TE_{\eta,j}.$$

Note that $\operatorname{Im}(E_{\rho,k}TE_{\eta,j}) \le I_{\rho,k}W_\rho$. Thus, the restriction of $E_{\rho,k}TE_{\eta,j}$ to $I_{\eta,j}W_\eta$ is an intertwining operator from $I_{\eta,j}W_\eta$ to $I_{\rho,k}W_\rho$. Therefore, by Corollary 1.2.2, we have

$$E_{\rho,k}TE_{\eta,j} = \begin{cases} 0, & \text{if } \eta \not\sim \rho, \\ \alpha^\rho_{k,j}T^\rho_{k,j} \text{ for some } \alpha^\rho_{k,j} \in \mathbb{C}, & \text{if } \rho \sim \eta. \end{cases}$$

This proves that the set (1.18) spans $\operatorname{End}_G(V)$.

It remains to prove that the set (1.18) is linearly independent. Suppose that

$$\sum_{\rho\in\widehat{G}_\sigma}\sum_{k,j=1}^{m_\rho} \alpha^\rho_{k,j}T^\rho_{k,j} = 0.$$

For $\rho \in \widehat{G}_\sigma$, choose a nonzero $v \in I_{\rho,j}W_\rho$. Since $T^\eta_{k,\ell}v = 0$ for $\eta \not\sim \rho$ or $\ell \ne j$, we have

$$\sum_{k=1}^{m_\rho} \alpha^\rho_{k,j}T^\rho_{k,j}v = 0$$

Since the $T^\rho_{k,j}v$, $1 \le k \le m_\rho$, belong to different summands in a decomposition of V into irreducible subrepresentations, they are linearly independent. Hence $\alpha^\rho_{k,j} = 0$ for all $1 \le k \le m_\rho$.

The fact that (1.19) is an isomorphism of algebras follows from (1.17). □

Corollary 1.2.11. *We have* $\dim \operatorname{End}_G(V) = \sum_{\rho\in\widehat{G}_\sigma} m_\rho^2$.

Definition 1.2.12. We say that the representation (σ, V) is *multiplicity free* if $m_\rho \le 1$ for all $\rho \in \widehat{G}$, or equivalently, if $m_\rho = 1$ for all $\rho \in \widehat{G}_\sigma$.

Corollary 1.2.13. *The representation* (σ, V) *is multiplicity free if and only if its commutant* $\operatorname{End}_G(V)$ *is commutative.*

Proof. This follows from Theorem 1.2.10 and the fact that the matrix algebra $M_{m,m}(\mathbb{C})$ is commutative if and only if $m = 1$. □

One of the nice properties of multiplicity free representations is that their decomposition into irreducible subrepresentations is unique, since it is the same as their decomposition into isotypic components.

Note that, for $\rho \in \widehat{G}_\sigma$,

$$E_\rho := \sum_{j=1}^{m_\rho} E_{\rho,j} = \sum_{j=1}^{m_\rho} T^\rho_{j,j}$$

is the projection from V onto the ρ-isotypic component $V_\rho = \bigoplus_{j=1}^{m_\rho} I_{\rho,j} W_\rho$. The projection E_ρ is called the *minimal central projection* or *minimal central idempotent* corresponding to ρ. The $E_{\rho,j}$ are called *minimal projections* or *minimal idempotents*.

Corollary 1.2.14. *The center $Z(\mathrm{End}_G(V))$ is isomorphic, as an algebra, to $\mathbb{C}^{\widehat{G}_\sigma}$. Furthermore, the minimal central projections $\{E_\rho : \rho \in \widehat{G}_\sigma\}$ are a basis for the center.*

Proof. This follows from Theorem 1.2.10 and Exercise 1.2.6. □

Exercises.

1.2.11. Let (σ, V) be a representation of G and $\rho \in \widehat{G}_\sigma$. Suppose $E_\rho = A + B$ for central idempotents $A, B \in \mathrm{End}_G(V)$ (i.e. $A^2 = A$, $B^2 = B$, and $A, B \in Z(\mathrm{End}_G(V))$). Prove that either $E_\rho = A$ (hence $B = 0$) or $E_\rho = B$ (hence $A = 0$). This justifies the term *minimal* central idempotent.

1.2.12. Let (σ, V) be a representation of G. Prove that $E_{\rho,j}$, $\rho \in \widehat{G}_\sigma$, $1 \leq j \leq m_\rho$, cannot be written as a sum of two nontrivial orthogonal idempotents. In other words, prove that if $E_{\rho,j} = A + B$ for $A, B \in \mathrm{End}_G(V)$ with $A^2 = A$, $B^2 = B$, and $AB = BA = 0$ (we say the idempotents A and B are orthogonal if $AB = BA = 0$), then $E_{\rho,j} = A$ (hence $B = 0$) or $E_{\rho,j} = B$ (hence $A = 0$). This justifies the term *minimal* idempotent (sometimes also called a *primitive idempotent*.)

1.2.13. Suppose (σ, V) and (η, U) are two representations of G with decompositions

$$V = \bigoplus_{\rho \in \widehat{G}_\sigma} W_\rho^{\oplus m_\rho} \quad \text{and} \quad U = \bigoplus_{\rho \in \widehat{G}_\eta} W_\rho^{\oplus n_\rho}$$

into irreducible subrepresentations. Prove that we have an isomorphism of vector spaces

$$\mathrm{Hom}_G(V, U) \cong \bigoplus_{\rho \in \widehat{G}_\sigma \cap \widehat{G}_\eta} M_{n_\rho, m_\rho}(\mathbb{C}).$$

1.2.5 Intertwiners as invariant elements

We have a canonical (i.e. basis independent) isomorphism of vector spaces

$$W' \otimes V \cong \mathrm{Hom}(W, V), \quad \varphi \otimes v \mapsto T_{\varphi,v}, \quad \text{where} \qquad (1.20)$$
$$T_{\varphi,v} w = \varphi(w) v, \quad \text{for all } w \in W.$$

(See Exercise 1.2.14.) It is important to remember here that not all elements of $W' \otimes V$ are simple tensors of the form $\varphi \otimes v$. However, it is enough to define a linear map on simple tensors, since we then extend it to all of the tensor product by linearity.

Suppose that (σ, V) and (ρ, W) are two representations of G. We define a representation of G on $\mathrm{Hom}(V, W)$ by

$$\eta(g)T = \sigma(g) T \rho\left(g^{-1}\right), \quad \text{for all } g \in G,\ T \in \mathrm{Hom}(W, V). \qquad (1.21)$$

Lemma 1.2.15. *The map (1.20) is an isomorphism from $\rho' \otimes \sigma$ to η.*

Proof. For $g \in G$, $\varphi \in W'$, $v \in V$, and $w \in W$, we have

$$\begin{aligned}
(\eta(g) T_{\varphi,v}) w &= \sigma(g) T_{\varphi,v} \rho\left(g^{-1}\right) w \\
&= \sigma(g) \varphi\left(\rho\left(g^{-1}\right) w\right) v \\
&= \varphi\left(\rho\left(g^{-1}\right) w\right) \sigma(g) v \\
&= (\rho'(g)\varphi)(w) \sigma(g) v \\
&= T_{\rho'(g)\varphi, \sigma(g)v} w.
\end{aligned}$$

Since

$$(\rho' \otimes \sigma)(g)(\varphi \otimes v) = (\rho'(g)\varphi) \otimes (\sigma(g) v),$$

this proves that the map (1.20) is an intertwiner. Since it is an isomorphism of vector spaces, it follows that it is an isomorphism of representations. □

Corollary 1.2.16. *We have*

$$\mathrm{Hom}_G(W, V) = \mathrm{Hom}(W, V)^G \cong \mathrm{Hom}_G(\iota_G, \rho' \otimes \sigma),$$

where the isomorphism is one of vector spaces and ι_G is the trivial representation of G.

Proof. For $T \in \mathrm{Hom}(W, V)$, we have

$$\begin{aligned}
T \in \mathrm{Hom}_G(W, V) &\iff \sigma(g) T = T \rho(g), \quad \text{for all } g \in G \\
&\iff \sigma(g) T \rho\left(g^{-1}\right) = T, \quad \text{for all } g \in G \\
&\iff \eta(g) T = T, \quad \text{for all } g \in G \\
&\iff T \in \mathrm{Hom}(W, V)^G.
\end{aligned}$$

Thus $\mathrm{Hom}_G(W, V) = \mathrm{Hom}(W, V)^G$.

Note that $\mathrm{Hom}(W, V)^G$ is precisely the ι_G-isotypic component of $\mathrm{Hom}(W, V)$ (Exercise 1.2.15). Thus,

$$\dim \mathrm{Hom}(W, V)^G = \dim \mathrm{Hom}(W, V)_{\iota_G} = \dim \mathrm{Hom}_G(\iota_G, \eta) = \dim \mathrm{Hom}_G(\iota_G, \rho' \otimes \sigma)$$

by Lemmas 1.2.5 and 1.2.15. Therefore, we have an isomorphism of vector spaces

$$\mathrm{Hom}(W, V)^G \cong \mathrm{Hom}_G(\iota_G, \rho' \otimes \sigma).$$

□

Exercises.

1.2.14. Prove that (1.20) is an isomorphism of vector spaces.

1.2.15. Suppose (σ, V) is a representation of G. Prove that V^G is the ι_G-isotypic component of V. (Recall that ι_G is the trivial representation of G.)

1.3 Characters and the projection formula

1.3.1 The trace

Suppose $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V. The *trace* is the linear map

$$\operatorname{tr}\colon \operatorname{End}(V) \to \mathbb{C}, \quad \operatorname{tr}(T) = \sum_{j=1}^{n} \langle Tv_j, v_j \rangle. \tag{1.22}$$

The trace is independent of the chosen basis. In fact, it is uniquely determined by the properties

(a) $\operatorname{tr}(TS) = \operatorname{tr}(ST)$ for all $S, T \in \operatorname{End}(V)$, and

(b) $\operatorname{tr}(I_V) = \dim V$.

(See Exercise 1.3.1.)

If $T \in \operatorname{End}(W)$ and $S \in \operatorname{End}(V)$, then $T \otimes S \in \operatorname{End}(W \otimes V)$. Choose orthonormal bases $\{v_1, \ldots, n\}$ and $\{w_1, \ldots, w_m\}$ for V and W, respectively. Then

$$\{w_i \otimes v_j : 1 \le i \le m,\ 1 \le j \le n\}$$

is an orthonormal basis for $W \otimes V$, and so

$$\begin{aligned}
\operatorname{tr}(T \otimes S) &= \sum_{i=1}^{m} \sum_{j=1}^{n} \langle (T \otimes S)(w_i \otimes v_j), w_i \otimes v_j \rangle_{W \otimes V} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \langle Tw_i, w_i \rangle_W \langle Sv_j, v_j \rangle_V \\
&= \operatorname{tr}(T) \operatorname{tr}(S).
\end{aligned}$$

Exercises.

1.3.1. Prove that the trace map is uniquely characterized by the fact that it is linear and satisfies the properties

(a) $\operatorname{tr}(TS) = \operatorname{tr}(ST)$ for all $S, T \in \operatorname{End}(V)$, and
(b) $\operatorname{tr}(I_V) = \dim V$.

More precisely, show that any map with these properties is given by (1.22) for any choice of orthonormal basis.

1.3.2 Central functions and characters

Definition 1.3.1 (Class function)**.** A function $f \in L(G)$ is *central* (or is a *class function*) if $f(gh) = f(hg)$ for all $g, h \in G$.

Lemma 1.3.2. *A function $f \in L(G)$ is a central if and only if it is conjugacy invariant:*

$$f\left(g^{-1}hg\right) = f(h), \quad \text{for all } g, h \in G.$$

In other words, a function is a central if and only if it is constant on each conjugacy class.

Proof. First suppose f is central. Then, for all $g, h \in G$, we have

$$f\left(g^{-1}hg\right) = f\left(hgg^{-1}\right) = f(h),$$

and so f is conjugacy invariant. On the other hand, if f is conjugacy invariant, then, for all $g, h \in G$, we have

$$f(gh) = f\left(ghgg^{-1}\right) = f(hg),$$

and so f is central. □

Lemma 1.3.2 justifies the terminology *class function* since one can think of such a function as a function from the set of conjugacy classes to $\mathbb{C}$.

Definition 1.3.3 (Character of a representation)**.** The *character* of a representation (σ, V) of G is the function

$$\chi^\sigma \colon G \to \mathbb{C}, \quad \chi^\sigma(g) = \operatorname{tr}(\sigma(g)).$$

In other words, $\chi^\sigma = \operatorname{tr} \circ \sigma$. A character of an irreducible representation is called an *irreducible character*.

The basic properties of characters are given in the following proposition.

Proposition 1.3.4. *Suppose (σ, V) and (ρ, W) are two representations of G.*

(a) $\chi^\sigma(1_G) = \dim V$

(b) $\chi^\sigma(g^{-1}) = \overline{\chi^\sigma(g)}$ *for all* $g \in G$.

(c) χ^σ *is a central function.*

(d) $\chi^{\sigma\oplus\rho} = \chi^\sigma + \chi^\rho$.

(e) *If* σ_i *is a representation of* G_i *for* $i = 1, 2$, *then*

$$\chi^{\sigma_1 \boxtimes \sigma_2}(g_1, g_2) = \chi^{\sigma_1}(g_1)\chi^{\sigma_2}(g_2), \quad \text{for all } (g_1, g_2) \in G_1 \times G_2.$$

(f) $\chi^{\sigma'}(g) = \overline{\chi^\sigma(g)}$ *for all* $g \in G$. *(Recall that* σ' *is the adjoint of* σ*.)*

(g) $\chi^{\sigma\otimes\rho} = \chi^\sigma\chi^\rho$.

(h) *If* η *is the representation of* G *on* $\operatorname{Hom}(V, W)$ *given by* (1.21), *then* $\chi^\eta = \overline{\chi^\rho}\chi^\sigma$.

Proof. Choose an orthonormal basis $\{v_1, \ldots, v_d\}$ for V.

(a) We have $\chi^\sigma(1_G) = \operatorname{tr}(\sigma(1_G)) = \operatorname{tr}(I_V) = \dim V$.

(b) For $g \in G$, we have

$$\chi^\sigma\left(g^{-1}\right) = \sum_{i=1}^d \left\langle \sigma\left(g^{-1}\right) v_i, v_i \right\rangle = \sum_{i=1}^d \langle v_i, \sigma(g) v_i\rangle = \sum_{i=1}^d \overline{\langle \sigma(g) v_i, v_i\rangle} = \overline{\chi^\sigma(g)}.$$

(c) For $g, h \in G$, we have

$$\chi^\sigma(gh) = \operatorname{tr}(\sigma(gh)) = \operatorname{tr}(\sigma(g)\sigma(h)) = \operatorname{tr}(\sigma(h)\sigma(g)) = \operatorname{tr}(\sigma(hg)) = \chi^\sigma(hg).$$

(d) For $g \in G$, we have

$$\chi^{\sigma\oplus\rho}(g) = \operatorname{tr}(\sigma(g) \oplus \rho(g)) = \operatorname{tr}(\sigma(g)) + \operatorname{tr}(\rho(g)) = \chi^\sigma(g) + \chi^\rho(g).$$

(e) For $(g_1, g_2) \in G_1 \times G_2$, we have

$$\chi^{\sigma_1\boxtimes\sigma_2}(g_1, g_2) = \operatorname{tr}(\sigma_1(g_1) \otimes \sigma_2(g_2)) = \operatorname{tr}(\sigma_1(g_1)) \operatorname{tr}(\sigma_2(g_2)) = \chi^{\sigma_1}(g_1)\chi^{\sigma_2}(g_2).$$

(f) Let $\{\theta_{v_1}, \ldots, \theta_{v_d}\}$ be the basis of V' dual to the chosen basis of V. Then

$$\begin{aligned}
\chi^{\sigma'}(g) &= \operatorname{tr}(\sigma'(g)) \\
&= \sum_{i=1}^d \langle \sigma'(g)\theta_{v_i}, \theta_{v_i}\rangle_{V'} \\
&\overset{(1.10)}{=} \sum_{i=1}^d \langle \theta_{\sigma(g)v_i}, \theta_{v_i}\rangle_{V'} \\
&\overset{(1.8)}{=} \sum_{i=1}^d \langle v_i, \sigma(g)v_i\rangle_V \\
&= \sum_{i=1}^d \overline{\langle \sigma(g)v_i, v_i\rangle_V} \\
&= \overline{\chi^\sigma(g)}.
\end{aligned}$$

(g) This follows from (e).

(h) By Lemma 1.2.15 and (f) and (g), we have

$$\chi^{\eta} = \chi^{\rho' \otimes \sigma} = \chi^{\rho'} \chi^{\sigma} = \overline{\chi^{\rho}} \chi^{\sigma}. \qquad \square$$

Exercises.

1.3.2. Compute the character of the sign representation ε of $\mathfrak{S}_n$ (Example 1.1.5).

1.3.3 Central projection formulas

If $E\colon V \to V$ is a projection, then

$$\dim(\operatorname{Im} E) = \operatorname{tr}(E). \tag{1.23}$$

(Exercise 1.3.3.) Recall that we may (and will) assume all representations are unitary by Lemma 1.1.1.

Lemma 1.3.5 (Basic projection formula)**.** *Suppose (σ, V) is a representation of G and $K \le G$. Then*

$$E_\sigma^K := \frac{1}{|K|} \sum_{k \in K} \sigma(k)$$

is the orthogonal projection of V onto V^K.

Proof. Let $E = E_\sigma^K$. For $v \in V$ and $g \in K$, we have

$$\sigma(g) E v = \frac{1}{|K|} \sum_{k \in K} \sigma(gk) v = Ev.$$

Thus $Ev \in V^K$. On the other hand, if $v \in V^K$, then

$$Ev = \frac{1}{|K|} \sum_{k \in K} \sigma(k) v = \frac{1}{|K|} \sum_{k \in K} v = v.$$

Thus E is a projection from V onto V^K. To see that E is orthogonal, we compute

$$E^* = \frac{1}{|K|} \sum_{k \in K} \sigma(k)^* = \frac{1}{|K|} \sum_{k \in K} \sigma\left(k^{-1}\right) = E. \qquad \square$$

Recall that ι_G is the trivial representation of a group G, with $\iota_G(g) = I_{\mathbb{C}}$ for all $g \in G$. Under the natural identification of $\operatorname{End}(\mathbb{C})$ with $\mathbb{C}$, ι_G corresponds to the element of $L(g)$ which is the constant function 1.

Corollary 1.3.6. *If (σ, V) is a representation of G and $K \leq G$, then*

$$\dim V^K = \frac{1}{|K|} \left\langle \chi^{\operatorname{Res}^G_K \sigma}, \iota_K \right\rangle_{L(K)}.$$

Proof. We have

$$\begin{aligned}
\dim V^K &= \operatorname{tr}\left(\frac{1}{|K|} \sum_{k \in K} \sigma(k) \right) && \text{((1.23) and Lemma 1.3.5)} \\
&= \frac{1}{|K|} \sum_{k \in K} \chi^\sigma(k) && \text{(linearity of tr and definition of } \chi^\sigma) \\
&= \frac{1}{|K|} \left\langle \chi^{\operatorname{Res}^G_K \sigma}, \iota_K \right\rangle_{L(K)} && \text{(definition (1.2) of scalar product on } L(K)).
\end{aligned}$$

□

Corollary 1.3.7 (Orthogonality of irreducible characters)**.** *Suppose (σ, V) and (ρ, W) are irreducible representations of G. Then*

$$\frac{1}{|G|} \langle \chi^\sigma, \chi^\rho \rangle_{L(G)} = \begin{cases} 1 & \text{if } \sigma \sim \rho, \\ 0 & \text{if } \sigma \not\sim \rho. \end{cases}$$

Proof. Let η be the representation of G on $\operatorname{Hom}(V, W)$ given by (1.21). Then

$$\begin{aligned}
\frac{1}{|G|} \langle \chi^\sigma, \chi^\rho \rangle_{L(G)} &= \frac{1}{|G|} \langle \overline{\chi^\rho} \chi^\sigma, \iota_G \rangle_{L(G)} \\
&= \frac{1}{|G|} \langle \chi^\eta, \iota_G \rangle_{L(G)} && \text{(Proposition 1.3.4(h))} \\
&= \dim \operatorname{Hom}(\rho, \sigma)^G && \text{(Corollary 1.3.6)} \\
&= \dim \operatorname{Hom}_G(\rho, \sigma) && \text{(Corollary 1.2.16)} \\
&= \begin{cases} 1 & \text{if } \sigma \sim \rho, \\ 0 & \text{if } \sigma \not\sim \rho \end{cases} && \text{(Schur's lemma).}
\end{aligned}$$

□

Corollary 1.3.8. *Let (σ, V) be a representation of G with multiplicities m_ρ, $\rho \in \widehat{G}$. Then*

(a) $m_\rho = \frac{1}{|G|} \langle \chi^\rho, \chi^\sigma \rangle_{L(G)}$ *for all* $\rho \in \widehat{G}$,

(b) $\frac{1}{|G|} \langle \chi^\sigma, \chi^\sigma \rangle_{L(G)} = \sum_{\rho \in \widehat{G}} m_\rho^2$, *and*

(c) $\frac{1}{|G|} \langle \chi^\sigma, \chi^\sigma \rangle_{L(G)} = 1$ *if and only if* σ *is irreducible.*

Proof. We leave the proof of this corollary as an exercise (Exercise 1.3.4). □

Note that Corollary 1.3.8 implies that σ is determined uniquely, up to equivalence, by its character. Since characters are readily computable, this is a very useful fact.

Lemma 1.3.9 (Fixed points character formula)**.** *Suppose G acts on a finite set X, and let $(\lambda, L(X))$ be the corresponding permutation representation (Example 1.1.3). Then*

$$\chi^\lambda(g) = |\{x \in X : gx = x\}|.$$

Proof. Recall that the Dirac functions, δ_x, $x \in X$, form an orthonormal basis for $L(X)$. So we compute

$$\begin{aligned}\chi^\lambda(g) &= \sum_{x\in X} \langle \lambda(g)\delta_x, \delta_x\rangle_{L(X)} \\ &= \sum_{x\in X} \langle \delta_{gx}, \delta_x\rangle_{L(X)} && \text{(by (1.4))} \\ &= |\{x \in X : gx = x\}|. \end{aligned}$$ □

Corollary 1.3.10. *The multiplicity of an irreducible representation (ρ, V_ρ) in the left regular representation $(\lambda, L(G))$ is equal to $d_\rho := \dim V_\rho$. In other words,*

$$L(G) \cong \bigoplus_{\rho\in\widehat{G}} V_\rho^{\oplus d_\rho} \quad \textit{(as representations of } G\textit{)}.$$

In particular $|G| = \sum_{\rho\in\widehat{G}} (\dim V_\rho)^2$.

Proof. By Lemma 1.3.9, we have

$$\chi^\lambda(g) = |\{h \in G : gh = h\}| = \begin{cases} |G| & \text{if } g = 1_G, \\ 0 & \text{otherwise,} \end{cases} = |G|\delta_{1_G}(g).$$

Thus, by Corollary 1.3.8(a), we have

$$m_\rho = \frac{1}{|G|}\langle \chi^\rho, \chi^\lambda\rangle_{L(G)} = \chi^\rho(1_G) = \dim V_\rho,$$

where the last equality follows from Proposition 1.3.4(a). □

Corollary 1.3.11. *Suppose G acts transitively on a finite set X. Choose $x_0 \in X$, and let*

$$K = \{g \in G : gx_0 = x_0\}$$

be the stabilizer *of x_0. Then the character of the permutation representation λ of G on X is given by*

$$\chi^\lambda(h) = \frac{|X|}{|\mathcal{C}|}|\mathcal{C} \cap K|, \quad \textit{for all } h \in G,$$

where $\mathcal{C}$ is the conjugacy class of G containing h.

Proof. Suppose $x \in X$. Since the action of G on X is transitive, there exists $s \in G$ such that $sx_0 = x$. Then we have a bijection

$$\{g \in \mathcal{C} : gx = x\} \to \{g \in \mathcal{C} : gx_0 = x_0\}, \quad g \mapsto s^{-1}gs.$$

Thus,

$$|\{g \in \mathcal{C} : gx = x\}| = |\{g \in \mathcal{C} : gx_0 = x_0\}| = |\mathcal{C} \cap K|. \tag{1.24}$$

Then, for $h \in G$, we have

$$\begin{aligned}
\chi^\lambda(h) &= |\{x \in X : hx = x\} && \text{(Lemma 1.3.9)} \\
&= \frac{1}{|\mathcal{C}|}|\{(x,g) \in X \times \mathcal{C} : gx = x\} \\
&= \frac{1}{|\mathcal{C}|}\sum_{x \in X} |\{g \in \mathcal{C} : gx = x\}| \\
&= \frac{|X|}{|\mathcal{C}|}|\mathcal{C} \cap K|. && \square
\end{aligned}$$

Definition 1.3.12 (Fourier transform)**.** Suppose (σ, V) is a representation of G and $f \in L(G)$. The operator

$$\sigma(f) := \sum_{g \in G} f(g)\sigma(g) \in \mathrm{End}(V)$$

is called the *Fourier transform* of f at σ.

Lemma 1.3.13 (Fourier transform of central functions)**.** *If $f \in L(G)$ is central and (ρ, W) is an irreducible representation, then*

$$\rho(f) = \frac{1}{d_\rho}\left\langle \chi^\rho, \bar{f} \right\rangle_{L(G)} I_W.$$

Proof. First we show that $\rho(f)$ is an intertwiner. For $h \in G$, we have

$$\begin{aligned}
\rho(f)\rho(h) &= \sum_{g \in G} f(g)\rho(gh) \\
&= \sum_{s \in G} f(sh^{-1})\rho(s) && (s = gh) \\
&= \sum_{s \in G} f(h^{-1}s)\rho(s) && (f \text{ is central}) \\
&= \sum_{g \in G} f(g)\rho(hg) && (g = h^{-1}s) \\
&= \rho(h)\rho(f).
\end{aligned}$$

Thus $\rho(f) \in \mathrm{End}_G(W)$. By Schur's lemma, there exists $c \in \mathbb{C}$ such that

$$\rho(f) = cI_W.$$

It remains to compute c. We have

$$cd_\rho = \mathrm{tr}(cI_W) = \mathrm{tr}(\rho(f)) = \mathrm{tr}\left(\sum_{g \in G} f(g)\rho(g)\right) = \sum_{g \in G} f(g)\chi^\rho(g) = \left\langle \chi^\rho, \bar{f} \right\rangle_{L(G)}.$$

Thus

$$c = \frac{1}{d_\rho} \left\langle \chi^\rho, \bar{f} \right\rangle_{L(G)},$$

as desired. □

Corollary 1.3.14. *If (σ, V) and (ρ, W) are irreducible representations, then*

$$\sigma\left(\overline{\chi^\rho}\right) = \begin{cases} \frac{|G|}{d_\sigma} I_V & \text{if } \sigma \sim \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 1.3.13 and Corollary 1.3.7, we have

$$\sigma\left(\overline{\chi^\rho}\right) = \frac{1}{d_\sigma} \left\langle \chi^\sigma, \chi^\rho \right\rangle I_V = \begin{cases} \frac{|G|}{d_\sigma} I_V & \text{if } \sigma \sim \rho, \\ 0 & \text{otherwise.} \end{cases}$$ □

Recall from Corollary 1.2.14 that, if (σ, V) is a representation of G, then the minimal central projections $E_\rho \in \mathrm{End}_G(V)$, $\rho \in \widehat{G}_\sigma$, form a basis for the center $Z(\mathrm{End}_G(V))$. We can now give an explicit expression for these minimal central projections.

Corollary 1.3.15 (Projection onto an isotypic component)**.** *Suppose (σ, V) is a representation of G. Then, for (ρ, W) an irreducible representation of G, the orthogonal projection onto the ρ-isotypic component of V is given by*

$$E_\rho = \frac{d_\rho}{|G|} \sigma\left(\overline{\chi^\rho}\right).$$

Proof. Let $\sigma = \bigoplus_{\eta \in \widehat{G}_\sigma} \bigoplus_{j=1}^{m_\eta} \eta_j$ be a decomposition of σ into irreducible subrepresentations, where $(\eta_j, W_{\eta,j})$ is a representation in the equivalence class η for each $1 \le j \le m_\eta$ (see (1.14)). Then

$$\begin{aligned} \frac{d_\rho}{|G|} \sigma\left(\overline{\chi^\rho}\right) &= \frac{d_\rho}{|G|} \sum_{\eta \in \widehat{G}_\sigma} \sum_{j=1}^{m_\eta} \eta_j\left(\overline{\chi^\rho}\right) \\ &= \sum_{j=1}^{m_\rho} E_{\rho,j} && \text{(Corollary 1.3.14)} \\ &= E_\rho. \end{aligned}$$ □

We would now like to determine the cardinality of $\widehat{G}$, that is, the number of irreducible nonequivalent representations of G. Note that, for $f \in L(G)$, we have

$$f \overset{(1.3)}{=} \sum_{g \in G} f(g) \delta_g \overset{(1.4)}{=} \sum_{g \in G} f(g) \lambda(g) \delta_{1_G} = \lambda(f) \delta_{1_G}. \tag{1.25}$$

Proposition 1.3.16. *The characters χ^ρ, $\rho \in \widehat{G}$, form an orthonormal basis for the space of central functions of G. In particular, $|\widehat{G}|$ is equal to the number of conjugacy classes of G.*

Proof. The elements

$$\chi^\rho, \quad \rho \in \widehat{G},$$

are orthogonal by Corollary 1.3.7. Since orthogonal vectors are linearly independent, it suffices to show that the characters span the space of central functions in $L(G)$. For this, it suffices to prove that the orthogonal complement to their span is zero.

Suppose $f \in L(G)$ is central and

$$\langle f, \chi^\rho \rangle_{L(G)} = 0, \quad \text{for all } \rho \in \widehat{G}.$$

Then

$$\begin{aligned} \lambda(f) &= \sum_{\rho \in \widehat{G}} d_\rho \rho(f) && \text{(Corollary 1.3.10)} \\ &= \sum_{\rho \in \widehat{G}} \langle \chi^\rho, \bar{f} \rangle_{L(G)} && \text{(Lemma 1.3.13)} \\ &= \sum_{\rho \in \widehat{G}} \langle f, \overline{\chi^\rho} \rangle_{L(G)} \\ &= \sum_{\rho \in \widehat{G}} \langle f, \chi^{\rho'} \rangle_{L(G)} && \text{(Proposition 1.3.4(f))} \\ &= 0. \end{aligned}$$

Therefore, $f = 0$ by (1.25).

If $\mathcal{C}$ denotes the set of all conjugacy classes of G, then the characteristic functions $\mathbf{1}_C$, $C \in \mathcal{C}$, defined by

$$\mathbf{1}_C(g) = \begin{cases} 1 & \text{if } g \in C, \\ 0 & \text{if } g \notin C, \end{cases}$$

form another basis for the space of central functions by Lemma 1.3.2. Thus, the dimension of this space is $|\mathcal{C}|$, and so $|\widehat{G}| = |\mathcal{C}|$. □

Example 1.3.17. If G is abelian, then every conjugacy class is a singleton, and all elements of $L(G)$ are central. It follows that $|\widehat{G}| = |G|$. For the cyclic group with n elements, we found the n inequivalent irreducible representations in Example 1.2.4.

The next theorem gives a precise relation between the representations of two groups and their product.

Theorem 1.3.18. *Suppose G_1 and G_2 are finite groups. Then we have a bijection of sets*

$$\widehat{G_1} \times \widehat{G_2} \to \widehat{G_1 \times G_2}, \quad (\rho_1, \rho_2) \mapsto \rho_1 \boxtimes \rho_2.$$

Proof. For $\rho_1, \sigma_1 \in \widehat{G_1}$ and $\rho_2, \sigma_2 \in \widehat{G_2}$, we have

$$\frac{1}{|G_1 \times G_2|} \left\langle \chi^{\rho_1 \boxtimes \rho_2}, \chi^{\sigma_1 \boxtimes \sigma_2} \right\rangle_{L(G_1 \times G_2)}$$

$$
\begin{aligned}
&= \frac{1}{|G_1 \times G_2|} \langle \chi^{\rho_1}\chi^{\rho_2}, \chi^{\sigma_1}\chi^{\sigma_2}\rangle_{L(G_1\times G_2)} && \text{(Proposition 1.3.4(e))}\\
&= \frac{1}{|G_1|}\langle \chi^{\rho_1}, \chi^{\sigma_1}\rangle_{L(G_1)} \frac{1}{|G_2|}\langle \chi^{\rho_2}, \chi^{\sigma_2}\rangle_{L(G_2)} && \text{(Exercise 1.3.5)}\\
&= \delta_{\rho_1,\sigma_1}\delta_{\rho_2,\sigma_2} && \text{(Corollary 1.3.7).}
\end{aligned}
$$

It then follows from Corollary 1.3.7 and Corollary 1.3.8(c) that

$$\rho_1 \boxtimes \rho_2, \quad \rho_1 \in \widehat{G_1},\ \rho_2 \in \widehat{G_2},$$

are pairwise inequivalent irreducible representations of $G_1 \times G_2$.

Now, by Proposition 1.3.16, $|\widehat{G_1 \times G_2}|$ is equal to the number of conjugacy classes of $G_1 \times G_2$, which is the product of the number of conjugacy classes of G_1 and the number of conjugacy classes of G_2 (Exercise 1.3.6.) By Proposition 1.3.16, this is equal to $|\widehat{G_1}| \cdot |\widehat{G_1}|$. So we have found *all* of the irreducible representations, completing the proof. □

Example 1.3.19. By the fundamental theorem of finite abelian groups (see, for example, [Jud, Th. 13.4]), every finite abelian group is a product of cyclic groups. Thus, Example 1.2.4 and Theorem 1.3.18 completely classify the irreducible representations of finite abelian groups, up to isomorphism.

Exercises.

1.3.3. Prove (1.23).

1.3.4. Prove Corollary 1.3.8.

1.3.5. Suppose X_1 and X_2 are finite sets. Any element $f \in L(X_1)$ can be viewed naturally as an element of $L(X_1 \times X_2)$ by setting $f(x_1, x_2) = f(x_1)$ for $(x_1, x_2) \in X_1 \times X_2$. Similarly, elements of $L(X_2)$ can also be viewed as elements of $L(X_1 \times X_2)$. Prove that

$$\langle f_1 f_2, g_1 g_2\rangle_{L(X_1\times X_2)} = \langle f_1, g_1\rangle_{L(X_1)}\langle f_2, g_2\rangle_{L(X_2)}, \quad \text{for all } f_1, g_1 \in L(X_1),\ f_2, g_2 \in L(X_2).$$

1.3.6. Suppose G_1 and G_2 are finite groups. Show that the conjugacy classes of $G_1 \times G_2$ are precisely the sets $C_1 \times C_2$, where C_i is a conjugacy class of G_i for $i = 1, 2$.

1.3.7. Consider the natural action of the symmetric group $\mathfrak{S}_n$ on the set $X = \{1, \dots, n\}$. Using the theory of characters discussed in this section, compute the multiplicity of the trivial representation in the corresponding permutation representation $(\lambda, L(X))$ of $\mathfrak{S}_n$.

1.4 Permutation representations

In this section, we study permutation representations (Example 1.1.3) in more detail.

1.4.1 Wielandt's lemma

Suppose G acts on a finite set X and let $(\lambda, L(X))$ denote the corresponding permutation representation (Example 1.1.3).

Define a product on $L(X,X)$ by:

$$(F_1F_2)(x,y) = \sum_{z\in X} F_1(x,z)F_2(z,y), \quad \text{for all } F_1, F_2 \in L(X\times X),\ x,y\in X. \tag{1.26}$$

Under this product and pointwise addition and scalar multiplication, $L(X\times X)$ is an algebra (Exercise 1.4.1). It may be viewed as the algebra of $X\times X$ matrices with coefficients in $\mathbb{C}$.

Lemma 1.4.1. *We have an isomorphism of algebras*

$$L(X\times X)\to \operatorname{End}(L(X)), \quad F\mapsto T_F, \quad \textit{where}$$
$$(T_Ff)(x) = \sum_{y\in X} F(x,y)f(y). \tag{1.27}$$

Proof. The proof of this lemma is left as an exercise (Exercise 1.4.1). □

The group G acts diagonally on $X\times X$:

$$g(x,y) = (gx, gy), \quad \text{for all } g\in G,\ x,y\in X.$$

Then we have the associated permutation representation on $L(X\times X)$. We can also view $\operatorname{End}(L(X)) = \operatorname{Hom}(L(X), L(X))$ as a representation of G as in (1.21).

Lemma 1.4.2. *The isomorphism of Lemma 1.4.1 is an intertwiner and hence an isomorphism of representations of G.*

Proof. Let λ_X denote the permutation representation on $L(X)$ and let $\lambda_{X\times X}$ denote the permutation representation on $L(X\times X)$. Furthermore, let η denote the representation of G on $\operatorname{End}(L(X))$ defined in (1.21). Then, for $F\in L(X\times X)$, $f\in L(X)$, $g\in G$, and $x\in X$, we have

$$\begin{aligned}
\big((\eta(g)T_F)f\big)(x) &= \Big(\lambda_X(g)T_F\big(\lambda_X(g^{-1})f\big)\Big)(x)\\
&= \lambda_X(g)\sum_{y\in X} F(x,y)(\lambda_X(g^{-1})f)(y)\\
&= \lambda_X(g)\sum_{y\in X} F(x,y)f(gy)\\
&= \sum_{y\in X} F(g^{-1}x,y)f(gy)\\
&= \sum_{z\in X} F(g^{-1}x,g^{-1}z)f(z) \qquad (z=gy)\\
&= \sum_{z\in X} \big(\lambda_{X\times X}(g)F\big)(x,z)f(z)\\
&= \big(T_{\lambda_{X\times X}(g)F}f\big)(x).
\end{aligned}$$

Thus, the isomorphism of Lemma 1.4.1 intertwines the G-action. □

Corollary 1.4.3. *We have* $\mathrm{End}_G(L(X)) \cong L(X \times X)^G$ *as algebras.*

Proof. By Lemma 1.4.2 and Corollary 1.2.16, we have isomorphisms of algebras

$$L(X \times X)^G \cong \mathrm{End}(L(X))^G = \mathrm{End}_G(L(X)).$$

□

Corollary 1.4.4 (Wielandt's lemma)**.** *Suppose* G *acts on a finite set* X*, and let* $\lambda = \bigoplus_{\rho \in \widehat{G}_\lambda} \rho^{\oplus m_\rho}$ *be the decomposition of the associated permutation representation into irreducibles. Then*

$$\sum_{\rho \in \widehat{G}_\lambda} m_\rho^2 = \textit{number of orbits of } G \textit{ on } X \times X.$$

Proof. Since the characteristic functions of the orbits of G on $X \times X$ form a basis of $L(X \times X)^G$, we have

$$\begin{aligned} \text{number of orbits of } G \text{ on } X \times X &= \dim L(X \times X)^G \\ &= \dim \mathrm{End}_G(L(X)) && \text{(Corollary 1.4.3)} \\ &= \sum_{\rho \in \widehat{G}_\lambda} m_\rho^2 && \text{(Corollary 1.2.11).} \end{aligned}$$

□

Example 1.4.5. Consider the natural action of the symmetric group $\mathfrak{S}_n$ on $X = \{1, 2, \dots, n\}$. We have an orthogonal direct sum decomposition (as representations)

$$\begin{gathered} L(X) = V_0 \oplus V_1, \quad \text{where} \\ V_0 = \{f \in L(X) : f(i) = f(j) \text{ for all } i, j \in X\}, \\ V_1 = \{f \in L(X) : \textstyle\sum_{j=1}^n f(j) = 0\}. \end{gathered} \tag{1.28}$$

On the other hand $\mathfrak{S}_n$ has precisely two orbits on $X \times X$, given by

$$\Omega_0 = \{(i,i) : i \in X\} \quad \text{and} \quad \Omega_1 = \{(i,j) : i, j \in X,\ i \neq j\}.$$

Thus, by Wiedlandt's lemma (Corollary 1.4.4), (1.28) is a decomposition into irreducible subrepresentations.

Exercises.

1.4.1. Show that $L(X, X)$ is an algebra under the product (1.26) and pointwise addition and scalar multiplication. Then prove Lemma 1.4.1.

1.4.2 ([CSST10, Ex. 1.4.2]). Suppose G acts on a finite set X. Show that the permutation reputation of G on $L(X \times X)$ is equivalent to the tensor product of the permutation representation $L(X)$ with itself. In other words, show that $L(X \times X) \cong L(X) \otimes L(X)$ as representations of G.

1.4.3 ([CSST10, Ex. 1.4.3]). Suppose G acts on finite sets X and Y. Show that

$$\operatorname{Hom}_G(L(X), L(Y)) \cong L(X \times Y)^G$$

as vector spaces.

1.4.4. Verify the details of Example 1.4.5. More precisely, do the following:

(a) Prove that V_0 and V_1 are $\mathfrak{S}_n$-invariant subspaces of $L(X)$.

(b) Prove that one has an orthogonal direct sum decomposition $L(X) = V_0 \oplus V_1$.

(c) Prove that Ω_0 and Ω_1 are $\mathfrak{S}_n$-orbits for the diagonal action of $\mathfrak{S}_n$ on $X \times X$.

1.4.5 ([CSST10, Ex. 1.4.6]). Suppose G acts transitively on a finite set X with at least two elements. As in Example 1.4.5, define

$$V_0 = \{f \in L(X) : f(x) = f(y) \text{ for all } x, y \in X\} \quad \text{and} \quad V_1 = \left\{ f \in L(X) : \sum_{x \in X} f(x) = 0 \right\}.$$

We say that the action of G on X is *doubly transitive* if

$$\forall\, x, y, z, u \in X \text{ such that } x \neq y \text{ and } z \neq u, \exists\, g \in G \text{ such that } g(x, y) = (z, u).$$

Prove that $L(X) = V_0 \oplus V_1$ is the decomposition of $L(X)$ into irreducible subrepresentations if and only if the action of G on X is doubly transitive.

1.4.6 ([CSST10, Ex. 1.4.7]). Suppose G acts on finite sets X and Y. For $\rho \in \widehat{G}$, let m_ρ and m'_ρ denote the multiplicities of ρ in $L(X)$ and $L(Y)$, respectively. Show that

$$\sum_{\rho \in \widehat{G}} m_\rho m'_\rho = \text{number of orbits of } G \text{ on } X \times Y.$$

1.4.2 Symmetric actions and Gelfand's lemma

A subset $A \subseteq X \times X$ is *symmetric* if

$$(x, y) \in A \implies (y, x) \in A.$$

The action of G on a set X is *symmetric* if all orbits of G on $X \times X$ are symmetric. A function $F \in L(X \times X)$ is *symmetric* if

$$F(x, y) = F(y, x), \quad \text{for all } x, y \in X.$$

Proposition 1.4.6 (Gelfand's lemma; symmetric case)**.** *Suppose that the action of G on a finite set X is symmetric. Then $\operatorname{End}_G(L(X))$ is commutative and the permutation representation of G on $L(X)$ is multiplicity free.*

Proof. Since the action of G is symmetric, any $F \in L(X \times X)^G$ is symmetric since it is constant on G-orbits. Therefore, for all $F_1, F_2 \in L(X \times X)^G$ and $x, y \in X$, we have

$$\begin{aligned}(F_1F_2)(x,y) &= \sum_{z\in X} F_1(x,z)F_2(z,y) \\ &= \sum_{z\in X} F_1(z,x)F_2(y,z) \\ &= (F_2F_1)(y,x) \\ &= (F_2F_1)(x,y) \qquad\qquad (F_2F_1 \in L(X\times X)^G \text{ by Corollary 1.4.3}).\end{aligned}$$

Hence $L(X \times X)^G$ is commutative. Thus, by Corollary 1.4.3, $\mathrm{End}_G(L(X))$ is commutative. Then the result follows from Corollary 1.2.13. □

Proposition 1.4.6 corresponds to the following fact for matrices: If $\mathcal{A}$ is a subalgebra of $M_{n,n}(\mathbb{C})$ consisting of symmetric matrices, then $\mathcal{A}$ is commutative, since

$$AB = A^T B^T = (BA)^T = BA, \quad \text{for all } A, B \in \mathcal{A}.$$

1.4.3 Frobenius reciprocity for permutation representations

In this section we will assume that G acts *transitively* on a finite set X. We fix $x_0 \in X$ and let

$$K = \{g \in G : gx_0 = x_0\}$$

denote its stabilizer. Then the map

$$G/K \to X, \quad gK \mapsto gx_0,$$

is a bijection of G-sets (i.e. it is a bijection of sets that commutes with the G-actions on X and on the set of right cosets G/K). So we may identify X with the space G/K of right cosets in G. We have

$$\sum_{x\in X} f(x) = \frac{1}{|K|}\sum_{g\in G} f(gx_0), \quad \text{for all } f \in L(X). \tag{1.29}$$

Definition 1.4.7 (Gelfand pair)**.** Suppose G acts transitively on X. We say that (G, K) is a *Gelfand pair* if the permutation representation $L(X)$ is multiplicity free. If the action of G on X is symmetric, we say that (G, K) is a *symmetric Gelfand pair*. (Note that, in this case, $L(X)$ is multiplicity free by Gelfand's lemma (Proposition 1.4.6).)

Example 1.4.8. Consider the natural action of the symmetric group $\mathfrak{S}_n$ on $\{1, 2, \ldots, n\}$. Fix $k \in \mathbb{Z}$, $0 \le k \le n/2$, and define

$$\Omega_{n-k,k} = \{A \subseteq \{1, 2, \ldots, n\} : |A| = k\}.$$

For $A \in \Omega_{n-k,k}$ and $\pi \in \mathfrak{S}_n$, we have

$$\pi A = \{\pi(j) : j \in A\} \in \Omega_{n-k,k},$$

so $\mathfrak{S}_n$ acts on the set $\Omega_{n-k,k}$.

Fix $A_0 \in \Omega_{n-k,k}$ and let K denote its stabilizer. Then

$$K \cong \mathfrak{S}_{n-k} \times \mathfrak{S}_k \quad \text{(as groups)},$$

where the first factor is the symmetric group on $A_0^{\complement} := \{1, 2, \ldots, n\} \setminus A_0$ and the second factor is the symmetric group on A_0. Since the action of $\mathfrak{S}_n$ on $\Omega_{n-k,k}$ is transitive, we may identify

$$\Omega_{n-k,k} = \mathfrak{S}_n / \left(\mathfrak{S}_{n-k} \times \mathfrak{S}_k\right).$$

Claim: Two elements (A, B) and (A', B') of $\Omega_{n-k,k} \times \Omega_{n-k,k}$ are in the same $\mathfrak{S}_n$-orbit (under the diagonal action) if and only if $|A \cap B| = |A' \cap B'|$.

Proof of claim: The "only if" part is clear. Suppose $|A \cap B| = |A' \cap B'|$. Consider the decomposition

$$\begin{aligned}\{1, 2, \ldots, n\} &= (A \cup B)^{\complement} \sqcup \big(A \setminus (A \cap B)\big) \sqcup \big(B \setminus (A \cap B)\big) \sqcup (A \cap B) \\ &= (A' \cup B')^{\complement} \sqcup \big(A' \setminus (A' \cap B')\big) \sqcup \big(B' \setminus (A' \cap B')\big) \sqcup (A' \cap B').\end{aligned}$$

We can choose $\pi \in \mathfrak{S}_n$ such that

- $\pi(A \cap B) = A' \cap B'$,
- $\pi(A \setminus (A \cap B)) = A' \setminus (A' \cap B')$, and
- $\pi(B \setminus (A \cap B)) = B' \setminus (A' \cap B')$,

since we know that each pair of sets involved have the same cardinality. It follows that $\pi(A, B) = (A', B')$. This proves the claim.

Now define

$$\Theta_j = \{(A, B) \in \Omega_{n-k,k} \times \Omega_{n-k,k} : |A \cap B| = j\}, \ 0 \le j \le k,$$

so that the decomposition of $\Omega_{n-k,k} \times \Omega_{n-k,k}$ into $\mathfrak{S}_n$-orbits is given by

$$\Omega_{n-k,k} \times \Omega_{n-k,k} = \bigsqcup_{j=0}^{k} \Theta_j.$$

(We use the fact that $k \le n/2$ here to conclude that the Θ_j are all nonempty.) Since $|A \cap B| = |B \cap A|$, every orbit is symmetric. So $(\mathfrak{S}_n, \mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ is a symmetric Gelfand pair.

Since there are precisely $k+1$ orbits of $\mathfrak{S}_n$ on $\Omega_{n-k,k} \times \Omega_{n-k,k}$, Wielandt's lemma (Corollary 1.4.4) implies that $L(\Omega_{n-k,k})$ decomposes into $k+1$ pairwise inequivalent irreducible $\mathfrak{S}_n$-representations. When $k = 1$, this recovers the result of Example 1.4.5.

Suppose (ρ, W) is an irreducible representation of G. Define $d_\rho = \dim W$ and suppose that W^K is nontrivial. For every $v \in W^K$, define a linear map

$$\mathcal{T}_v \colon W \to L(X), \quad (\mathcal{T}_v u)(g x_0) = \sqrt{\frac{d_\rho}{|X|}} \, \langle u, \rho(g) v \rangle_W, \quad \text{for all } g \in G, \ u \in W. \tag{1.30}$$

This is defined on all of X since the action of G on X is transitive. Furthermore, if $g, h \in G$ and $gx_0 = hx_0$, then $g^{-1}h \in K$, and thus

$$\begin{aligned}
(\mathcal{T}_v u)(hx_0) &= \sqrt{\frac{d_\rho}{|X|}}\,\langle u, \rho(h)v\rangle_W \\
&= \sqrt{\frac{d_\rho}{|X|}}\,\langle u, \rho(g)\rho(g^{-1}h)v\rangle_W \\
&= \sqrt{\frac{d_\rho}{|X|}}\,\langle u, \rho(g)v\rangle_W \qquad \text{(since } v \in V^K\text{)} \\
&= (\mathcal{T}_v u)(gx_0).
\end{aligned}$$

Hence $\mathcal{T}_v u$ is well defined.

Theorem 1.4.9 (Frobenius reciprocity for permutation representations)**.** *With notation as above, we have the following.*

(a) $\mathcal{T}_v \in \mathrm{Hom}_G(W, L(X))$ *for all* $v \in W^K$.

(b) (Orthogonality relations) *For all* $v, u \in W^K$ *and* $w, z \in W$, *we have*

$$\langle \mathcal{T}_u w, \mathcal{T}_v z\rangle_{L(X)} = \langle w, z\rangle_W \langle v, u\rangle_W.$$

(c) The map

$$W^K \to \mathrm{Hom}_G(W, L(X)), \quad v \mapsto \mathcal{T}_v, \tag{1.31}$$

is an antilinear isomorphism. In particular, the multiplicity of ρ *in the permutation representation* $L(X)$ *is equal to* $\dim W^K$.

Proof. (a) For $g, h \in G$ and $w \in W$, we have

$$\begin{aligned}
(\lambda(g)\mathcal{T}_v w)(hx_0) &= (\mathcal{T}_v w)\left(g^{-1}hx_0\right) \\
&= \sqrt{\frac{d_\rho}{|X|}}\,\langle w, \rho(g^{-1})\rho(h)v\rangle_W \\
&= \sqrt{\frac{d_\rho}{|X|}}\,\langle \rho(g)w, \rho(h)v\rangle_W \\
&= (\mathcal{T}_v \rho(g) w)(hx_0).
\end{aligned}$$

Hence $\lambda(g)\mathcal{T}_v = \mathcal{T}_v\rho(g)$ for all $g \in G$, and so $\mathcal{T}_v \in \mathrm{Hom}_G(W, L(X))$.

(b) For $u, v \in W^K$, define a linear map

$$R_{u,v} \colon W \to W, \quad R_{u,v}w = \langle w, u\rangle_W v, \quad \text{for all } w \in W.$$

Choosing an orthonormal basis $\{w_1, \dots, w_{d_\rho}\}$ for W, we compute

$$\mathrm{tr}(R_{u,v}) = \sum_{j=1}^{d_\rho} \langle R_{u,v}w_j, w_j\rangle_W = \sum_{j=1}^{d_\rho} \langle w_j, u\rangle_W \langle v, w_j\rangle_W = \left\langle v, \sum_{j=1}^{d_\rho} \langle u, w_j\rangle w_j \right\rangle_W = \langle v, u\rangle_W.$$

While $R_{u,v} \in \mathrm{End}(W)$, in general $R_{u,v}$ is not an element of $\mathrm{End}_G(W)$. However, we can project it onto $\mathrm{End}_G(W) = \mathrm{End}(W)^G$ (see Corollary 1.2.16) to get

$$R := \frac{1}{|G|} \sum_{g \in G} \rho(g) R_{u,v} \rho(g^{-1}) \in \mathrm{End}_G(W). \tag{1.32}$$

(Here we use Lemma 1.3.5 and the action on $\mathrm{End}(W)$ given by (1.21).) Since W is irreducible, by Schur's lemma (Lemma 1.2.1) we have $R = cI_W$ for some $c \in \mathbb{C}$. Taking the trace of both sides of (1.32), we have

$$\begin{aligned} cd_\rho &= \mathrm{tr}(cI_W) \\ &= \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}\left(\rho(g) R_{u,v} \rho(g^{-1})\right) \\ &= \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(R_{u,v}) \\ &= \langle v, u \rangle_W. \end{aligned}$$

Thus $c = \frac{1}{d_\rho} \langle v, u \rangle_W$, and so

$$R = \frac{1}{d_\rho} \langle v, u \rangle_W I_W. \tag{1.33}$$

Then, for $w, z \in W$, we have

$$\begin{aligned} \langle \mathcal{T}_u w, \mathcal{T}_v z \rangle_{L(X)} &= \sum_{x \in X} (\mathcal{T}_u w)(x) \overline{(\mathcal{T}_v z)(x)} \\ &= \frac{1}{|K|} \sum_{g \in G} (\mathcal{T}_u w)(gx_0) \overline{(\mathcal{T}_v z)(gx_0)} && \text{(by (1.29))} \\ &= \frac{d_\rho}{|G|} \sum_{g \in G} \langle w, \rho(g) u \rangle_W \overline{\langle z, \rho(g) v \rangle_W} && \text{(since } |K| \cdot |X| = |G|) \\ &= \frac{d_\rho}{|G|} \sum_{g \in G} \langle \langle w, \rho(g) u \rangle_W \rho(g) v, z \rangle_W \\ &= \frac{d_\rho}{|G|} \sum_{g \in G} \left\langle \rho(g) \langle \rho(g^{-1}) w, u \rangle_W v, z \right\rangle_W \\ &= \frac{d_\rho}{|G|} \sum_{g \in G} \langle \rho(g) R_{u,v} \rho(g^{-1}) w, z \rangle_W \\ &= d_\rho \langle Rw, z \rangle_W && \text{(by (1.32))} \\ &= \langle w, z \rangle_W \langle v, u \rangle_W && \text{(by (1.33)).} \end{aligned}$$

(c) The map (1.31) is antilinear since the bilinear form is antilinear in the second argument. We now show that it is bijective. Fix $T \in \mathrm{Hom}_G(W, L(X))$ and consider the composition of linear maps

$$W \xrightarrow{T} L(X) \xrightarrow{f \mapsto f(x_0)} \mathbb{C}, \quad u \mapsto (Tu)(x_0),$$

As discussed in Section 1.1.5, this implies that there exists a *unique* $v \in W$ such that

$$(Tu)(x_0) = \langle u, v \rangle_W, \quad \text{for all } u \in W.$$

Then

$$\begin{aligned} (Tu)(gx_0) &= \left(\lambda(g^{-1})Tu\right)(x_0) \\ &= \left(T\rho(g^{-1})u\right)(x_0) \qquad\qquad (T \in \mathrm{Hom}_G(W, L(X))) \\ &= \langle \rho(g^{-1})u, v \rangle_W \\ &= \langle u, \rho(g)v \rangle_W, \end{aligned}$$

which implies that

$$T = \sqrt{\frac{|X|}{d_\rho}}\, \mathcal{T}_v.$$

We also have $v \in W^K$ since, for $k \in K$,

$$\langle u, \rho(k)v \rangle_W = (Tu)(kx_0) = (Tu)(x_0) = \langle u, v \rangle_W, \quad \text{for all } u \in W,$$

and so $\rho(k)v = v$ by the nondegeneracy of the scalar product. Since the vector v was uniquely determined by T, we see that (1.31) is bijective.

Finally, by Lemma 1.2.5, the multiplicity of ρ in the permutation representation on $L(X)$ is equal to

$$\dim \mathrm{Hom}_G(W, L(X)) = \dim W^K. \qquad \square$$

Corollary 1.4.10. *The pair (G, K) is a Gelfand pair if and only if $\dim W^K \le 1$ for every irreducible G-representation W. In particular, (G, K) is a Gelfand pair if and only if*

$$\dim W^K = 1 \iff W \textit{ is a subrepresentation of } L(X).$$

Exercises.

1.4.7 ([CSST10, Ex. 1.4.11]). (a) Show that, if $0 \le h \le k \le n/2$, then $\mathfrak{S}_n$ has precisely $h + 1$ orbits on $\Omega_{n-k,k} \times \Omega_{n-h,h}$.

(b) Suppose that $L(\Omega_{n-k,k}) = \bigoplus_{j=0}^{k} V_{k,j}$ is the decomposition of $L(\Omega_{n-k,k})$ into irreducible subrepresentations (see Example 1.4.8). Use part (a) and Exercise 1.4.6 to show that it is possible to number the representations $V_{k,0}, \ldots, V_{k,j}$ in such a way that $V_{h,j} \cong V_{k,j}$ (as representations) for all $j = 0, 1, \ldots, h$ and $0 \le h \le k \le n/2$. *Hint*: Every subrepresentation of $L(\Omega_{n-h,h})$ is also a subrepresentation of $L(\Omega_{n-k,k})$.

1.4.4 The structure of the commutant of a permutation representation

The goal of this subsection is to give an explicit form for the operators $T^{\rho}_{k,j}$ defined in (1.16). Recall that, by Theorem 1.2.10, these operators give a basis for the commutant $\operatorname{End}_G(V)$.

As in Section 1.4.3, we assume that G acts transitively on a finite set X. We fix $x_0 \in X$, and let K be the stabilizer of x_0, so that we can identify X with G/K.

For $\rho \in \widehat{G}$, let m_ρ be the multiplicity of ρ in the permutation representation $(\lambda, L(X))$. For each $\rho \in \widehat{G}_\lambda$, by Theorem 1.4.9, we have $\dim W_\rho^K = m_\rho$, so we can choose an orthonormal basis

$$\{v_1^\rho, \dots, v_{m_\rho}^\rho\}$$

for W_ρ^K. Let

$$\mathcal{T}_j^\rho := \mathcal{T}_{v_j^\rho} \in \operatorname{Hom}_G(W_\rho, L(X)), \quad \rho \in \widehat{G}_\lambda,\ 1 \le j \le m_\rho,$$

be the intertwiners defined in (1.30) (which are intertwiners by Theorem 1.4.9(a)), so that

$$\left(\mathcal{T}_j^\rho u\right)(gx_0) = \sqrt{\frac{d_\rho}{|X|}}\left\langle u, \rho(g) v_j^\rho \right\rangle_{W_\rho}, \quad \text{for all } g \in G, u \in W_\rho. \tag{1.34}$$

Recall that if U and V are unitary spaces, an *isometric immersion* of U into V is a linear map $T \colon U \to V$ such that

$$\langle Tu_1, Tu_2 \rangle_V = \langle u_1, u_2 \rangle_U, \quad \text{for all } u_1, u_2 \in U.$$

The term *immersion* comes from the fact that such a map is necessarily injective (Exercise 1.4.8).

Lemma 1.4.11. *We have that*

$$L(X) = \bigoplus_{\rho \in \widehat{G}_\lambda} \bigoplus_{j=1}^{m_\rho} \mathcal{T}_j^\rho W_\rho \tag{1.35}$$

is an orthogonal decomposition of $L(X)$ into irreducible subrepresentations. Furthermore, every $\mathcal{T}_j^\rho$ is an isometric immersion of W_ρ into $L(X)$.

Proof. It follows from Theorem 1.4.9(b) that $\bigoplus_{\rho \in \widehat{G}_\lambda} \bigoplus_{j=1}^{m_\rho} \mathcal{T}_j^\rho W_\rho$ is an orthogonal decomposition (i.e. the summands are orthogonal) and that the $\mathcal{T}_j^\rho$ are isometric immersions. Then, since

$$\dim \left(\bigoplus_{\rho \in \widehat{G}_\lambda} \bigoplus_{j=1}^{m_\rho} \mathcal{T}_j^\rho W_\rho \right) = \sum_{\rho \in \widehat{G}_\lambda} \sum_{j=1}^{m_\rho} \dim \mathcal{T}_j^\rho W_\rho = \sum_{\rho \in \widehat{G}_\lambda} \sum_{j=1}^{m_\rho} \dim W_\rho = \dim L(X),$$

we see that we have the equality (1.35). □

Now, for $\rho \in \widehat{G}_\lambda$ and $1 \le i, j \le m_\rho$, define

$$\phi^\rho_{i,j} \in L(X \times X), \quad \phi^\rho_{i,j}(gx_0, hx_0) = \frac{d_\rho}{|X|} \left\langle \rho(h)v^\rho_j, \rho(g)v^\rho_i \right\rangle_{W_\rho}, \quad g, h \in G. \tag{1.36}$$

Note that $\phi^\rho_{i,j}$ is well defined since v^ρ_i and v^ρ_j are K-invariant. Moreover, $\phi^\rho_{i,j} \in L(X \times X)^G$, since, for all $s, g, h \in G$, we have

$$\phi^\rho_{i,j}(sgx_0, shx_0) = \frac{d_\rho}{|X|} \left\langle \rho(s)\rho(h)v^\rho_j, \rho(s)\rho(g)v^\rho_i \right\rangle_{W_\rho} = \phi^\rho_{i,j}(gx_0, hx_0),$$

since ρ is unitary.

Therefore, as in (1.27) and Corollary 1.4.3, we can define

$$\Phi^\rho_{i,j} \in \mathrm{End}_G(L(X)),$$
$$\left(\Phi^\rho_{i,j} f\right)(x) = \sum_{y \in X} \phi^\rho_{i,j}(x, y) f(y), \quad \text{for all } f \in L(X),\ x \in X.$$

Lemma 1.4.12. *For all $g \in G$ and $f \in L(X)$, we have*

$$\Phi^\rho_{i,j} f = \frac{\sqrt{d_\rho}}{\sqrt{|X|}\,|K|} \mathcal{T}^\rho_i \left(\sum_{h \in G} \rho(h) f(hx_0) v^\rho_j \right).$$

Proof. We compute

$$\begin{aligned}
\left(\Phi^\rho_{i,j} f\right)(gx_0) &= \sum_{y \in X} \phi^\rho_{i,j}(gx_0, y) f(y) \\
&= \frac{1}{|K|} \sum_{h \in G} \phi^\rho_{i,j}(gx_0, hx_0) f(hx_0) \\
&= \frac{d_\rho}{|K|\,|X|} \sum_{h \in G} \left\langle \rho(h)v^\rho_j, \rho(g)v^\rho_i \right\rangle_{W_\rho} f(hx_0) && \text{(by (1.36))} \\
&= \frac{\sqrt{d_\rho}}{\sqrt{|X|}\,|K|} \sum_{h \in G} \left(\mathcal{T}^\rho_i \rho(h) v^\rho_j\right)(gx_0) \cdot f(hx_0) && \text{(by (1.34))} \\
&= \frac{\sqrt{d_\rho}}{\sqrt{|X|}\,|K|} \mathcal{T}^\rho_i \left(\sum_{h \in G} \rho(h) f(hx_0) v^\rho_j \right)(gx_0).
\end{aligned}$$

□

Using Lemma 1.4.12, we can now show that the $\Phi^\rho_{i,j}$ are the operators $T^\rho_{i,j}$ of Theorem 1.2.10.

Theorem 1.4.13. *For all $\sigma, \rho \in \widehat{G}_\lambda$, $1 \le i, j \le m_\rho$, and $1 \le s, r, \le m_\sigma$, we have*

(a) $\mathrm{Im}\, \Phi^\rho_{i,j} = \mathcal{T}^\rho_i W_\rho$,

(b) $\Phi^\rho_{i,j} \Phi^\sigma_{s,r} = \delta_{j,s} \delta_{\rho,\sigma} \Phi^\rho_{i,r}$.

(c) $\mathrm{Ker}\, \Phi^\rho_{i,j} = L(X) \ominus \mathcal{T}^\rho_j W_\rho$ *(here $\ominus$ means we remove the summand $\mathcal{T}^\rho_j W_\rho$ from $L(X)$), and*

Proof. (a) Since W_ρ is irreducible, the G-invariant subspace generated by $v_j^\rho \in W_\rho$ is all of W_ρ. Thus, it follows from Lemma 1.4.12 that $\operatorname{Im}\Phi_{i,j}^\rho = \mathcal{T}_i^\rho W_\rho$.

(b) For all $g, h \in G$, we compute the product:

$$\begin{aligned}
&\left(\phi_{i,j}^\rho \phi_{s,r}^\sigma\right)(gx_0, hx_0)\\
&\quad = \frac{1}{|K|}\sum_{t\in G}\phi_{i,j}^\rho(gx_0, tx_0)\phi_{s,r}^\sigma(tx_0, hx_0)\\
&\quad = \frac{d_\rho d_\sigma}{|X|^2|K|}\sum_{t\in G}\langle\rho(t)v_j^\rho, \rho(g)v_i^\rho\rangle_{W_\rho}\langle\sigma(h)v_r^\sigma, \sigma(t)v_s^\sigma\rangle_{W_\sigma} && \text{(by (1.36))}\\
&\quad = \frac{\sqrt{d_\rho d_\sigma}}{|X|\,|K|}\sum_{t\in G}\overline{\left(\mathcal{T}_j^\rho\rho(g)v_i^\rho\right)(tx_0)}\left(\mathcal{T}_s^\sigma\rho(h)v_r^\sigma\right)(tx_0) && \text{(by (1.34))}\\
&\quad = \frac{\sqrt{d_\rho d_\sigma}}{|X|}\left\langle\mathcal{T}_s^\sigma\rho(h)v_r^\sigma, \mathcal{T}_j^\rho\rho(g)v_i^\rho\right\rangle_{L(X)}\\
&\quad = \delta_{\sigma,\rho}\frac{d_\rho}{|X|}\left\langle v_j^\rho, v_s^\rho\right\rangle_{W_\rho}\langle\rho(h)v_r^\rho, \rho(g)v_i^\rho\rangle_{W_\rho} && \text{(Th. 1.4.9(b), Lem. 1.4.11)}\\
&\quad = \delta_{\sigma,\rho}\delta_{j,s}\phi_{i,r}^\rho(gx_0, hx_0) && \text{(by (1.36))}.
\end{aligned}$$

Thus $\phi_{i,j}^\rho\phi_{s,r}^\sigma = \delta_{\sigma,\rho}\delta_{j,s}\phi_{i,r}^\rho\phi_{i,r}^\rho$. So the result follows from Lemma 1.4.1.

(c) Suppose $\sigma \in \widehat{G}_\lambda$ and $1 \le k \le m_\sigma$ such that $\sigma \ne \rho$ or that $\sigma = \rho$, $j \ne k$. Then, by parts (a) and (b), we have

$$\Phi_{i,j}^\rho\mathcal{T}_k^\sigma W_\sigma = \operatorname{Im}\Phi_{i,j}^\rho\Phi_{k,k}^\sigma = 0.$$

Since $\Phi_{i,j}^\rho$ is not the zero map by part (a), it must be nonzero on the remaining irreducible summand $\mathcal{T}_j^\rho W_\rho$. □

Corollary 1.4.14. *(a) The map $\Phi_{i,i}^\rho$ is the orthogonal projection onto $\mathcal{T}_i^\rho W_\rho$.*

(b) The map $\sum_{i=1}^{m_\rho}\Phi_{i,i}^\rho$ is the orthogonal projection onto the ρ-isotypic component.

Exercises.

1.4.8. Prove that every isometric immersion is injective.

1.4.9. Prove that

$$\left(\Phi_{i,j}^\rho f\right)(gx_0) = \sqrt{\frac{d_\rho}{|X|}}\left\langle f, \mathcal{T}_j^\rho\rho(g)v_i^\rho\right\rangle_{L(X)}, \quad \text{for all } g \in G,\ f \in L(X).$$

Hint: Follow the method of Lemma 1.4.12.

1.5 The group algebra and the Fourier transform

In this section we consider the special case of Section 1.4 where $X = G$. That is, we study the left regular representation $(\lambda, L(G))$ of a the finite group G.

1.5.1 The group algebra

Recall the left regular representation λ and right regular representation ρ of G on $L(G)$ from Example 1.1.4. In the language of Section 1.4, we take $x_0 = 1_G$, so that $K = \{1_G\}$.

We define a *convolution product* $*$ on $L(G)$ by

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(gh^{-1}) f_2(h), \quad \text{for all } f_1, f_2 \in L(G),\ g \in G. \tag{1.37}$$

One can show that $L(G)$ is a unital algebra with this product (Exercise 1.5.1). It is called the *group algebra* (or the *convolution algebra*) of G. The unit of this algebra is δ_{1_G}.

Remark 1.5.1. The group algebra of G is sometimes defined to be set of formal linear combinations

$$\mathbb{C}G = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{C} \text{ for all } g \in G \right\},$$

with multiplication given by

$$\left(\sum_{g \in G} \alpha_g g \right) \left(\sum_{h \in G} \beta_h h \right) = \sum_{g,h \in G} (\alpha_g \beta_h)(gh).$$

We have an obvious isomorphism

$$\mathbb{C}G \xrightarrow{\cong} L(G), \quad \sum_{g \in G} \alpha_g g \mapsto \sum_{g \in G} \alpha_g \delta_g,$$

so this is essentially the same construction.

Note that we can also express the convolution product as

$$(f_1 * f_2)(g) = \sum_{s,t \in G : st = g} f_1(s) f_2(t) = \sum_{h \in G} f_1(h) f_2(h^{-1}g), \quad \text{for all } f_1, f_2 \in L(G),\ g \in G.$$

It follows that

$$(f_1 * f_2) = \sum_{h \in G} f_1(h) \lambda(h) f_2.$$

For $g \in G$ and $f \in L(G)$, we have

$$\delta_g * f = \sum_{h \in G} \delta_g(h) \lambda(h) f_2 = \lambda(g) f. \tag{1.38}$$

Similarly,

$$f * \delta_g = \rho(g^{-1}) f.$$

In particular,

$$\delta_g * \delta_h = \delta_{gh}, \quad \text{for all } g, h \in G. \tag{1.39}$$

It follows that

$$\rho(g)\lambda(h) = \lambda(h)\rho(g), \quad \text{for all } g, h \in G.$$

In other words, the left and right regular actions commute.

Lemma 1.5.2. *The map*

$$L(G) \to L(G), \quad \psi \mapsto \check{\psi}, \quad \textit{where}$$
$$\check{\psi}(g) = \overline{\psi(g^{-1})}, \quad \textit{for all } g \in G,$$

is an involution on the algebra $L(G)$.

Proof. For $\psi_1, \psi_2 \in L(G)$ and $g \in G$, we have

$$\begin{aligned} \left(\check{\psi}_1 * \check{\psi}_2\right)(g) &= \sum_{s \in G} \check{\psi}_1(gs)\check{\psi}_2(s^{-1}) \\ &= \sum_{s \in G} \overline{\psi_1(s^{-1}g^{-1})\psi_2(s)} \\ &= \overline{(\psi_2 * \psi_1)(g^{-1})} \\ &= (\psi_2 * \psi_1)\check{}\,(g). \end{aligned}$$

We leave it as an exercise to verify the remaining properties of an involution (Exercise 1.5.2). □

Remark 1.5.3. Note that

$$\begin{aligned} f \in Z(L(G)) &\iff \delta_g * f = f * \delta_g, \quad \text{for all } g \in G \\ &\iff \lambda(g)f = \rho(g^{-1})f, \quad \text{for all } g \in G \\ &\iff f(g^{-1}h) = f(hg^{-1}), \quad \text{for all } g, h \in G \\ &\iff f \text{ is central.} \end{aligned}$$

Proposition 1.5.4. *The map*

$$L(G) \to \mathrm{End}_G(L(G)), \quad \psi \mapsto T_\psi, \quad \textit{where}$$
$$T_\psi f = f * \psi, \quad \textit{for all } f \in L(G),$$

is a $*$*-anti-isomorphism of algebras.*

Proof. We leave it as an exercise (Exercise 1.5.4) to show that the linear map

$$\mathrm{End}_G(L(G)) \to L(G), \quad T \mapsto T(\delta_{1_G}). \tag{1.40}$$

is inverse to the linear map $\psi \mapsto T_\psi$.

For $\psi_1, \psi_2, f \in L(G)$, we have

$$T_{\psi_1}(T_{\psi_2} f) = (f * \psi_2) * \psi_1 = f * (\psi_2 * \psi_1) = T_{\psi_2 * \psi_1} f.$$

Thus $T_{\psi_1}T_{\psi_2} = T_{\psi_2 * \psi_1}$, and so the map $\psi \mapsto T_\psi$ is an anti-multiplicative. Furthermore, for $f_1, f_2, \psi \in L(G)$, we have

$$\begin{aligned}
\langle T_\psi f_1, f_2\rangle &= \langle f_1 * \psi, f_2\rangle \\
&= \sum_{g\in G}(f_1 * \psi)(g)\,\overline{f_2(g)} \\
&= \sum_{g\in G}\sum_{s\in G} f_1(gs)\psi(s^{-1})\,\overline{f_2(g)} \\
&= \sum_{t\in G}\sum_{s\in G} f_1(t)\psi(s^{-1})\,\overline{f_2(ts^{-1})} \qquad (t = gs) \\
&= \sum_{t\in G}\sum_{s\in G} f_1(t)\,\overline{\check{\psi}(s) f_2(ts^{-1})} \\
&= \sum_{t\in G} f_1(t)\,\overline{(f_2 * \check{\psi})(t)} \\
&= \left\langle f_1, T_{\check{\psi}} f_2\right\rangle_{L(G)}.
\end{aligned}$$

Hence $(T_\psi)^* = T_{\check{\psi}}$. □

For every $\rho \in \widehat{G}$, fix an orthonormal basis

$$\{v_1^\rho, v_2^\rho, \ldots, v_{d_\rho}^\rho\}$$

for the representation space W_ρ. Then define the corresponding unitary matrix coefficients

$$\varphi_{i,j}^\rho(g) = \left\langle \rho(g)v_j^\rho, v_i^\rho\right\rangle_{W_\rho}. \tag{1.41}$$

We deduced some basic properties of matrix coefficients in Lemma 1.1.12. In the case that the representations are irreducible, we have the following additional properties.

Lemma 1.5.5. *Let ρ and σ be two irreducible representations of G.*

(a) (Orthogonality relations) $\left\langle \varphi_{i,j}^\rho, \varphi_{s,t}^\sigma\right\rangle_{L(G)} = \dfrac{|G|}{d_\rho}\delta_{\rho,\sigma}\delta_{i,s}\delta_{j,t}$.

(b) $\varphi_{i,j}^\rho * \varphi_{s,t}^\sigma = \dfrac{|G|}{d_\rho}\delta_{\rho,\sigma}\delta_{j,s}\varphi_{i,t}^\rho$.

Proof. (a) We have

$$\begin{aligned}
\langle \varphi_{i,j}^\rho, \varphi_{s,t}^\sigma\rangle_{L(G)} &= \sum_{g\in G} \varphi_{i,j}^\rho(g)\overline{\varphi_{s,t}^\sigma(g)} \\
&= \sum_{g\in G} \left\langle \rho(g)v_j^\rho, v_i^\rho\right\rangle_{W_\rho} \overline{\langle \rho(g)v_t^\sigma, v_s^\sigma\rangle_{W_\rho}} \\
&= \sqrt{\frac{|G|}{d_\rho}}\sqrt{\frac{|G|}{d_\sigma}} \sum_{g\in G} \overline{(\mathcal{T}_{v_j^\rho}^\rho v_i^\rho)(g)}(\mathcal{T}_{v_t^\sigma}^\sigma v_s^\sigma)(g) \qquad \text{(see (1.30))}
\end{aligned}$$

$$= \sqrt{\frac{|G|}{d_\rho}}\sqrt{\frac{|G|}{d_\sigma}} \left\langle \mathcal{T}^{\sigma}_{v^{\sigma}_t} v^{\sigma}_s, \mathcal{T}^{\rho}_{v^{\rho}_j} v^{\rho}_i \right\rangle_{L(G)}$$

$$= \frac{|G|}{d_\rho}\delta_{\rho,\sigma}\delta_{i,s}\delta_{j,t} \qquad \text{(Theorem 1.4.9(b))}.$$

(b) We have

$$\begin{aligned}
\left(\varphi^{\rho}_{i,j} * \varphi^{\sigma}_{s,t}\right)(g) &= \sum_{h\in G} \varphi^{\rho}_{i,j}(gh)\varphi^{\sigma}_{s,t}(h^{-1}) \\
&= \sum_{h\in G}\sum_{k=1}^{d_\rho} \varphi^{\rho}_{i,k}(g)\varphi^{\rho}_{k,j}(h)\overline{\varphi^{\sigma}_{t,s}(h)} && \text{(Lemma 1.1.12)} \\
&= \frac{|G|}{d_\rho}\sum_{k=1}^{d_\rho} \varphi^{\rho}_{i,k}\delta_{\rho,\sigma}\delta_{k,t}\delta_{j,s} && \text{(part (a))} \\
&= \frac{|G|}{d_\rho}\delta_{\rho,\sigma}\delta_{j,s}\varphi^{\rho}_{i,t}(g).
\end{aligned}$$

□

Corollary 1.5.6. *The set*

$$\left\{\varphi^{\rho}_{i,j} : \rho \in \widehat{G},\ 1 \le i,j \le d_\rho\right\}$$

is an orthogonal basis for $L(G)$.

Proof. The given elements are orthogonal, hence linearly independent, by Lemma 1.5.5. By Corollary 1.3.10, $\dim L(G) = \sum_{\rho\in\widehat{G}} d_\rho^2$, which is equal to the cardinality of the given set. □

Exercises.

1.5.1. Prove that $L(G)$ is an algebra under the product $*$ defined in (1.37) and componentwise addition and scalar multiplication. Prove that it is commutative if and only if G is abelian.

1.5.2. Complete the proof of Lemma 1.5.2.

1.5.3. Suppose $\mathcal{A}$ is a unital algebra and let $\mathcal{A}^{\mathrm{op}}$ denote the *opposite algebra*. Precisely, $\mathcal{A}^{\mathrm{op}}$ is equal to $\mathcal{A}$ as a vector space, but the multiplication in $\mathcal{A}^{\mathrm{op}}$ is given by

$$a \cdot b = ba, \quad a, b \in \mathcal{A},$$

where $\cdot$ denotes the multiplication in $\mathcal{A}^{\mathrm{op}}$ and juxtaposition denotes the multiplication in $\mathcal{A}$. Note that an algebra anti-homomorphism $\mathcal{A} \to \mathcal{B}$ is the same as an algebra homomorphism $\mathcal{A} \to \mathcal{B}^{\mathrm{op}}$.

Let $\operatorname{End}\mathcal{A}$ denote the algebra of linear maps from $\mathcal{A}$ to $\mathcal{A}$, and define

$$\operatorname{End}_{\mathcal{A}}\mathcal{A} = \{f \in \operatorname{End}\mathcal{A} : f(ab) = af(b)\ \forall\, a, b \in \mathcal{A}\}.$$

This is a subalgebra of $\operatorname{End} \mathcal{A}$. (In fact, $\operatorname{End}_{\mathcal{A}} \mathcal{A}$ is the algebra of $\mathcal{A}$-module endormorphisms of $\mathcal{A}$, considered as a module over itself.)

(a) Show that $\operatorname{End}_{\mathcal{A}} \mathcal{A} \cong \mathcal{A}^{\mathrm{op}}$ as algebras.

(b) In the case that $\mathcal{A}$ is the group algebra $L(G)$, prove that $\operatorname{End}_{L(G)} L(G) = \operatorname{End}_G(L(G))$. Here, $\operatorname{End}_G(L(G))$ refers to the commutant of the left regular representation $(\lambda, L(G))$.

(c) Using the above facts, give another proof that the map of Proposition 1.5.4 is an anti-isomorphism of algebras. (You are not asked to give a new proof that the map respects the involutions.)

1.5.4. Show that (1.40) is inverse to the map $\psi \mapsto T_\psi$ of Proposition 1.5.4.

1.5.2 The Fourier transform

Suppose (σ, V) is a representation of G and recall the definition (Definition 1.3.12) of the Fourier transform of elements of $L(G)$ at σ. For $f_1, f_2 \in L(G)$, we have

$$\begin{aligned} \sigma(f_1 * f_2) &= \sum_{g \in G} (f_1 * f_2)(g)\sigma(g) \\ &= \sum_{g,h \in G} f_1(gh) f_2(h^{-1})\sigma(gh)\sigma(h^{-1}) \\ &= \sigma(f_1)\sigma(f_2). \end{aligned} \tag{1.42}$$

The *Fourier transform* is the algebra homomorphism

$$\mathcal{F}\colon L(G) \to \mathcal{A}(G) := \bigoplus_{\rho \in \widehat{G}} \operatorname{End}(W_\rho), \quad f \mapsto \bigoplus_{\rho \in \widehat{G}} \rho(f).$$

We define a scalar product on $\mathcal{A}(G)$ by

$$\left\langle \bigoplus_{\rho \in \widehat{G}} T_\rho, \bigoplus_{\sigma \in \widehat{G}} S_\sigma \right\rangle = \frac{1}{|G|} \sum_{\rho \in \widehat{G}} d_\rho \operatorname{tr}\left(T_\rho S_\rho^*\right). \tag{1.43}$$

(See Exercise 1.5.5.)

Recall that $\{v_1^\rho, \dots, v_{d_\rho}^\rho\}$ is an orthonormal basis for W_ρ. For $\rho \in \widehat{G}$ and $1 \le i, j \le d_\rho$, define

$$T_{i,j}^\rho \in \mathcal{A}(G), \quad T_{i,j}^\rho w = \delta_{\rho,\sigma} \left\langle w, v_j^\rho \right\rangle_{W_\rho} v_i^\rho, \quad w \in W_\sigma,\ \sigma \in \widehat{G}.$$

Thus, the map $T_{i,j}^\rho$ sends v_j^ρ to v_i^ρ and v_k^σ to zero for all $\sigma \neq \rho$, $1 \le k \le d_\sigma$ or $\sigma = \rho$, $k \neq ij$.

Theorem 1.5.7. *The Fourier transform $\mathcal{F}$ is an isometric $*$-isomorphism between the algebras $L(G)$ and $\mathcal{A}(G)$. Furthermore,*

$$\mathcal{F}\,\overline{\varphi_{i,j}^\rho} = \frac{|G|}{d_\rho} T_{i,j}^\rho, \quad \text{for all } \rho \in \widehat{G},\ 1 \le i, j \le d_\rho. \tag{1.44}$$

Proof. A proof of this statement can be found in [CSST10, Th. 1.5.11]. □

Let $\mathcal{B}' \subseteq \mathcal{A}(G)$ be the subalgebra consisting of elements that are diagonal in the bases $\{v_1^\rho, \dots, v_{d_\rho}^\rho\}$. Then $\mathcal{B}'$ is a maximal commutative subalgebra of $\mathcal{A}(G)$. (See Exercise 1.5.6.) Define

$$\mathcal{B} = \mathcal{F}^{-1}(\mathcal{B}'),$$

so that $\mathcal{B}$ is a maximal commutative subalgebra of $L(G)$. Note that $\mathcal{B}$ depends on our choice of bases for the W_ρ, $\rho \in \widehat{G}$.

The *primitive idempotent* associated with the vector v_j^ρ is the group algebra element

$$e_j^\rho := \frac{d_\rho}{|G|}\overline{\varphi_{j,j}^\rho} \in L(G). \tag{1.45}$$

Thus

$$e_j^\rho(g) = \frac{d_\rho}{|G|}\overline{\varphi_{j,j}^\rho(g)} = \frac{d_\rho}{|G|}\overline{\langle \rho(g)w_j^\rho, w_j^\rho\rangle_{W_\rho}}, \quad \text{for all } g \in G. \tag{1.46}$$

Proposition 1.5.8. *(a) The set*

$$\left\{e_j^\rho : \rho \in \widehat{G},\ j = 1, 2, \dots, d_\rho\right\}$$

is a vector space basis for $\mathcal{B}$*.*

(b) For all $\rho, \sigma \in \widehat{G}$, $1 \le j \le d_\rho$, $1 \le i \le d_\sigma$, *we have*

$$e_j^\rho * e_i^\sigma = \delta_{\rho,\sigma}\delta_{j,i}e_j^\rho.$$

(So the e_j^ρ *are orthogonal idempotents.)*

(c) For all $\rho, \sigma \in \widehat{G}$, $1 \le j \le d_\rho$, $1 \le i \le d_\sigma$, *we have*

$$\sigma(e_j^\rho)v_i^\sigma = \delta_{\rho,\sigma}\delta_{j,i}v_j^\rho.$$

In particular, $\rho(e_j^\rho)\colon W_\rho \to W_\rho$ *is the orthogonal projection onto* $\mathbb{C}v_j^\rho$*.*

(d) If $f \in \mathcal{B}$ *satisfies*

$$\rho(f)v_j^\rho = \lambda_j^\rho v_j^\rho, \quad \textit{for all } \rho \in \widehat{G},\ 1 \le j \le d_\rho,$$

for some $\lambda_j^\rho \in \mathbb{C}$*, then*

$$f = \sum_{\rho\in\widehat{G}}\sum_{j=1}^{d_\rho}\lambda_j^\rho e_j^\rho \qquad \text{(Fourier inversion formula in } \mathcal{B}\text{)}$$

and

$$f * e_j^\rho = \lambda_j^\rho e_j^\rho.$$

Proof. (a) By (1.44) and (1.45), we have

$$\mathcal{F}(e_j^\rho) = T_{j,j}^\rho,$$

which is the diagonal matrix acting as 1 on v_j^ρ and as 0 on v_i^σ for $\sigma \ne \rho$ or $i \ne j$. Since such matrices form a basis for $\mathcal{B}'$, it follows from Theorem 1.5.7 that the e_j^ρ form a basis for $\mathcal{B}$.

(b) We have

$$\begin{aligned} e_j^\rho * e_i^\sigma &= \frac{d_\rho d_\sigma}{|G|^2}\overline{\varphi_{j,j}^\rho} * \overline{\varphi_{i,i}^\sigma} \\ &= \frac{d^\rho}{|G|}\delta_{\rho,\sigma}\delta_{j,i}\overline{\varphi_{j,j}^\rho} && \text{(Lemma 1.5.5(b))} \\ &= \delta_{\rho,\sigma}\delta_{j,i}e_j^\rho. \end{aligned}$$

(c) We have

$$\bigoplus_{\sigma\in\widehat{G}} \sigma(e_j^\rho)v_i^\sigma = \left(\mathcal{F}e_j^\rho\right) v_i^\sigma = T_{j,j}^\rho v_i^\sigma = \delta_{\rho,\sigma}\langle v_i^\rho, v_j^\rho\rangle_{W_\rho} v_j^\rho = \delta_{\rho,\sigma}\delta_{i,j}v_j^\rho \in W_\rho.$$

Comparing W_σ components, we see that

$$\sigma(e_j^\rho)v_i^\sigma = \delta_{\rho,\sigma}\delta_{j,i}v_j^\rho.$$

(d) Suppose $f \in \mathcal{B}$ satisfies

$$\rho(f)v_j^\rho = \lambda_j^\rho v_j^\rho, \quad \text{for all } \rho \in \widehat{G},\ 1 \le j \le d_\rho,$$

for some $\lambda_j^\rho \in \mathbb{C}$. By part (a), we have

$$f = \sum_{\rho\in\widehat{G}}\sum_{j=1}^{d_\rho} \mu_j^\rho e_j^\rho$$

for some $\mu_j^\rho \in \mathbb{C}$. Then, for $\sigma \in \widehat{G}$ and $1 \le i \le d_\sigma$, we have

$$\begin{aligned} \lambda_i^\sigma v_i^\sigma &= \sigma(f)v_i^\sigma \\ &= \sum_{\rho\in\widehat{G}}\sum_{j=1}^{d_\rho} \mu_j^\rho \sigma(e_j^\rho)v_i^\sigma \\ &= \sum_{\rho\in\widehat{G}}\sum_{j=1}^{d_\rho} \mu_j^\rho \delta_{\rho,\sigma}\delta_{j,i}v_j^\rho && \text{(part (c))} \\ &= \mu_i^\sigma v_i^\sigma. \end{aligned}$$

Thus $\mu_i^\sigma = \lambda_i^\sigma$, as desired.

Finally, we have

$$f * e_j^\rho = \left(\sum_{\sigma\in\widehat{G}}\sum_{i=1}^{d_\sigma} \lambda_i^\sigma e_i^\sigma\right) * e_j^\rho = \lambda_j^\rho e_j^\rho,$$

by part (b).

□

Exercises.

1.5.5. Prove that (1.43) defines a scalar product on $\mathcal{A}(G)$.

1.5.6. Let $\mathcal{A}$ be the subalgebra of $M_{n,n}(\mathbb{C})$ consisting of diagonal matrices. Prove that $\mathcal{A}$ is a maximal commutative subalgebra of $M_{n,n}(\mathbb{C})$. In other words, prove that if $\mathcal{B}$ is a commutative subalgebra of $M_{n,n}(\mathbb{C})$ containing $\mathcal{A}$, then $\mathcal{B} = \mathcal{A}$.

1.5.7 ([CSST10, Ex. 1.5.18]). (a) Consider the decomposition (1.35) in the case of the group algebra (so $X = G$ with the left regular action). Show that $\psi \in L(G)$ belongs to $\mathcal{T}_j^\rho W_\rho$ if and only if $\psi * e_i^\sigma = \delta_{\sigma,\rho}\delta_{i,j}\psi$ for all $\sigma \in \widehat{G}$ and $1 \le i \le d_\sigma$.

(b) Show that $f \in L(G)$ belongs to $\mathcal{B}$ if and only if each subspace $\mathcal{T}_j^\rho W_\rho$ is an eigenspace for the associated convolution operator $T_f \colon \psi \mapsto \psi * f$; moreover, if $f \in \sum_{\rho\in\widehat{G}} \sum_{j=1}^{d_\rho} \lambda_j^\rho e_j^\rho$, then the eigenvalue of T_f corresponding to $\mathcal{T}_j^\rho W_\rho$ is λ_j^ρ.

1.5.3 Algebras of bi-K-invariant functions

Suppose G acts transitively on a finite set X. Fix $x_0 \in X$ and let K be the stabilizer of x_0. Then, as before, we can identify X with G/K.

Let S be a set of representatives of the double cosets $K\backslash G/K$ of K in G. In other words, $\{KsK : s \in S\}$ are the equivalences classes for the relation on G defined by

$$g \sim h \iff \exists\, k_1, k_2 \in K \text{ such that } g = k_1hk_2$$

So we have

$$G = \bigsqcup_{s\in S} KsK.$$

For $s \in S$, define

$$\Omega_s := Ksx_0 = KsKx_0 = \{ksx_0 : k \in K\},$$
$$\Theta_s := G(x_0, sx_0) = \{(gx_0, gsx_0) : g \in G\} \subseteq X \times X.$$

Lemma 1.5.9. *(a) $X = \bigsqcup_{s\in S} \Omega_s$ is the decomposition of X into K-orbits.*

(b) $X \times X = \bigsqcup_{s\in S} \Theta_s$ is the decomposition of $X \times X$ into G-orbits (under the diagonal action).

Proof. (a) For $g \in G$, Kgx_0 is the K-orbit of gx_0. Furthermore, for $h \in G$, we have

$$\begin{aligned} Kgx_0 = Khx_0 &\iff \exists\, k_1 \in K \text{ such that } gx_0 = k_1hx_0 \\ &\iff \exists\, k_1 \in K \text{ such that } (k_1h)^{-1}gx_0 = x_0 \\ &\iff \exists\, k_1, k_2 \in K \text{ such that } (k_1h)^{-1}g = k_2 \\ &\iff \exists\, k_1, k_2 \in K \text{ such that } g = k_1hk_2 \\ &\iff g \in KhK. \end{aligned}$$

Since S is a set of representatives of $K\backslash G/K$, it follows that Ksx_0, $s \in S$, are the K-orbits on X.

(b) Suppose $x, y \in X$. Then $x = g_0 x_0$ for some $g_0 \in G$. Since the action of G on X is transitive, we can choose $g \in G$ such that $gx_0 = g_0^{-1} y$. Then

$$G(x, y) = G(g_0 x_0, g_0 g x_0) = G(x_0, gx_0).$$

Thus, every G-orbit on $X \times X$ is of the form $G(x_0, gx_0)$ for some $g \in G$.

Furthermore, for $g, h \in G$, we have

$$\begin{aligned} G(x_0, gx_0) = G(x_0, hx_0) &\iff \exists\, k_1 \in K \text{ such that } (x_0, gx_0) = (x_0, k_1 h x_0) \\ &\iff \exists\, k_1, k_2 \in K \text{ such that } g = k_1 h k_2. \end{aligned}$$ □

Definition 1.5.10 (Left, right and bi-K-invariant functions)**.** Let K be a subgroup of a finite group G.

- $f \in L(G)$ is *right K-invariant* if $f(gk) = f(g)$ for all $g \in G$, $k \in K$.
- $f \in L(G)$ is *left K-invariant* if $f(kg) = f(g)$ for all $g \in G$, $k \in K$.
- $f \in L(G)$ is *bi-K-invariant* if it is both left and right K-invariant.

We let $L(G/K)$ denote the subspace of right K-invariant functions and let $L(K\backslash G/K)$ denote the subspace of bi-K-invariant functions. (Note that this notation corresponds to the fact that right K-invariant functions can be viewed as functions on G/K and vice versa, and similarly for bi-K-invariant functions.)

It is straightforward to verify (Exercise 1.5.9) that the space $L(G/K)$ is a left ideal in $L(G)$ and hence, by (1.38), it is invariant under the left regular action. It follows that we can restrict the permutation representation of G on $L(G)$ to obtain a representation of G on $L(G/K)$. Similarly, $L(K\backslash G)$ is a right ideal in $L(G)$. It follows that $L(K\backslash G/K)$ is a subalgebra of $L(G)$.

Theorem 1.5.11. *(a) The map*

$$L(X) \to L(G/K), \quad f \mapsto \tilde{f}, \quad \textit{where}$$
$$\tilde{f}(g) = f(gx_0), \quad \textit{for all } g \in G,$$

is an isomorphism of G-representations.

(b) The map

$$L(X \times X)^G \to L(K\backslash G/K), \quad F \mapsto \tilde{F}, \quad \textit{where}$$
$$\tilde{F}(g) = \frac{1}{|K|} F(x_0, gx_0), \quad \textit{for all } g \in G,$$

is an isomorphism of algebras.

Proof. (a) This follows immediately from the isomorphism of G-sets $X \cong G/K$.

(b) The map $F \mapsto \tilde{F}$ is clearly linear. It then follows from Exercise 1.5.8 that it is an isomorphism of vector spaces. Now suppose $F_1, F_2 \in L(X \times X)^G$ and let

$$F(x,y) = \sum_{z \in X} F_1(x,z)F_2(z,y), \quad \text{for all } x, y \in X,$$

so that F is the product of F_1 and F_2 in $L(X \times X)^G$. Then, for all $g \in G$,

$$\begin{aligned}
\tilde{F}(g) &= \frac{1}{|K|} F(x_0, gx_0) \\
&= \frac{1}{|K|^2} \sum_{h \in G} F_1(x_0, hx_0)F_2(hx_0, gx_0) && \text{(note we sum over } G\text{)} \\
&= \frac{1}{|K|^2} \sum_{h \in G} F_1(x_0, hx_0)F_2(x_0, h^{-1}gx_0) && \left(\text{since } F_2 \in L(X \times X)^G\right) \\
&= \sum_{h \in G} \tilde{F}_1(h)\tilde{F}_2(h^{-1}g) \\
&= \left(\tilde{F}_1 * \tilde{F}_2\right)(g).
\end{aligned}$$

□

By Theorem 1.5.11 and Lemma 1.5.9 we have

$$L(X \times X)^G \cong L(K \backslash G / K) \cong L(X)^K,$$

where the first isomorphism is one of algebras, and the second is one of vector spaces. Using the second isomorphism, one can endow $L(X)^K$ with the structure of an algebra.

Corollary 1.5.12. *We have that (G, K) is a Gelfand pair if and only if the algebra $L(K \backslash G / K)$ is commutative.*

Proof. We have

$$\begin{aligned}
(G,K) \text{ is a Gelfand pair} &\iff L(X) \text{ is multiplicity free} && \text{(definition of Gelfand pair)} \\
&\iff \mathrm{End}_G(L(X)) \text{ is commutative} && \text{(Corollary 1.2.13)} \\
&\iff L(X \times X)^G \text{ is commutative} && \text{(Corollary 1.4.3)} \\
&\iff L(K \backslash G / K) \text{ is commutative} && \text{(Theorem 1.5.11(b))}.
\end{aligned}$$

□

We are now able to prove the general form of Gelfand's lemma (see Proposition 1.4.6).

Proposition 1.5.13 (Gelfand's lemma)**.** *Suppose G is a finite group and $K \leq G$ is a subgroup. Furthermore, suppose there exists an automorphism τ of G such that $g^{-1} \in K\tau(g)K$ for all $g \in G$. Then (G, K) is a Gelfand pair.*

Proof. If $f \in L(K \backslash G / K)$, then $f(\tau(g)) = f(g^{-1})$ for all $g \in G$. Thus, for all $f_1, f_2 \in L(K \backslash G / K)$ and $g \in G$, we have

$$(f_1 * f_2)(\tau(g)) = \sum_{s \in G} f_1(\tau(g)s) f_2\left(s^{-1}\right)$$

$$
\begin{aligned}
&= \sum_{h\in G} f_1(\tau(gh))f_2(\tau(h^{-1})) \qquad\qquad (h=\tau^{-1}(s))\\
&= \sum_{h\in G} f_1\left((gh)^{-1}\right) f_2(h)\\
&= \sum_{h\in G} f_2(h) f_1\left(h^{-1}g^{-1}\right)\\
&= (f_2 * f_1)\left(g^{-1}\right)\\
&= (f_2 * f_1)(\tau(g)),
\end{aligned}
$$

and so $L(K\backslash G/K)$ is commutative. Then the result follows from Corollary 1.5.12. □

We say that (G, K) is a *weakly symmetric* Gelfand pair if it satisfies the hypotheses of Proposition 1.5.13.

Example 1.5.14. The group $G \times G$ acts on G by

$$(g_1, g_2)\cdot g = g_1 g g_2^{-1}, \quad \text{for all } g_1, g_2, g \in G.$$

The stabilizer of 1_G is the diagonal subgroup

$$\tilde{G} = \{(g,g) : g \in G\} \le G \times G,$$

and thus $G \cong (G\times G)/\tilde{G}$ as G-sets. Now consider the flip automorphism

$$\tau\colon G\times G \to G\times G, \quad \tau(g_1,g_2) = (g_2,g_1), \quad \text{for all } (g_1,g_2)\in G\times G.$$

Then, for all $g_1, g_2 \in G$, we have

$$(g_1,g_2)^{-1} = \left(g_1^{-1}, g_2^{-1}\right) = \left(g_1^{-1}, g_1^{-1}\right)(g_2,g_1)\left(g_2^{-1}, g_2^{-1}\right) \in \tilde{G}\tau(g_1,g_2)\tilde{G}.$$

Therefore, $(G\times G, \tilde{G})$ is a weakly symmetric Gelfand pair.

Exercises.

1.5.8 ([CSST10, Ex. 1.5.20]). For every orbit Ω of K on X, set

$$\Theta_\Omega = \{(gx_0, gx) : g \in G,\ x \in \Omega\}.$$

Show that the map $\Omega \mapsto \Theta_\Omega$ is a bijection from the set of K-orbits on X to the set of G-orbits on $X \times X$.

1.5.9. Verify that $L(G/K)$ is a left ideal in $L(G)$ and that $L(K\backslash G)$ is a right ideal in $L(G)$. Then prove that $L(K\backslash G/K)$ is a subalgebra of $L(G)$.

1.5.10 ([CSST10, Ex. 1.5.25]). Show that (G, K) is a symmetric Gelfand pair if and only if $g^{-1} \in KgK$ for all $g \in G$. Note that this corresponds to the case $\tau = I_G$ in Proposition 1.5.13.

1.5.11 ([CSST10, Ex. 1.5.27]). A group G is *ambivalent* if g^{-1} is conjugate to g for every $g \in G$. We adopt the notation of Example 1.5.14. Show that the Gelfand pair $(G \times G, \tilde{G})$ is symmetric if and only if G is ambivalent.

1.6 Induced representations

1.6.1 Definitions and examples

Let K be a subgroup of G and let (ρ, V) be a representation of K. Consider the action of K on G given by

$$k \cdot g = gk^{-1}, \quad k \in K,\ g \in G.$$

Definition 1.6.1 (Induced representation)**.** The *representation induced* by (ρ, V) is the G-representation (σ, Z) defined by

$$Z = \mathrm{Fun}_K(G, V) := \left\{ f \colon G \to V : f(gk) = \rho(k^{-1})f(g),\ \text{for all } g \in G,\ k \in K \right\} \tag{1.47}$$

and

$$(\sigma(g_1)f)(g_2) = f\left(g_1^{-1}g_2\right), \quad \text{for all } g_1, g_2 \in G,\ f \in Z. \tag{1.48}$$

(Note that Z is a vector space under pointwise addition and scalar multiplication, and that $\sigma(g)f \in Z$ for all $g \in G$ and $f \in Z$.) We introduce the notation

$$\mathrm{Ind}_K^G \rho = \sigma \quad \text{and} \quad \mathrm{Ind}_K^G V = Z.$$

One can also give another description of induced representations. Fix a set S of representatives for the set G/K of right cosets of K in G, so that

$$G = \bigsqcup_{s \in S} sK. \tag{1.49}$$

For every $v \in V$, define

$$f_v \colon G \to V, \quad f_v(g) = \begin{cases} \rho\left(g^{-1}\right) v & \text{if } g \in K, \\ 0 & \text{if } g \notin K. \end{cases} \tag{1.50}$$

Then $f_v \in Z$. Furthermore, the space

$$\tilde{V} = \{f_v \colon v \in V\} \tag{1.51}$$

is a K-invariant subspace of Z and the map

$$V \to \tilde{V}, \quad v \mapsto f_v,$$

is an isomorphism of K-representations. (See Exercise 1.6.1.)

We claim that

$$Z = \bigoplus_{s\in S} \sigma(s)\tilde{V} \tag{1.52}$$

as vector spaces. Indeed, for every $f \in Z$ and $s \in S$, let $v_s = f(s)$. Then we have

$$f = \sum_{s\in S} \sigma(s) f_{v_s} \tag{1.53}$$

and this is the unique way to write f as a sum of elements of $\sigma(s)\tilde{V}$, $s \in S$. (See Exercise 1.6.1.) Furthermore, for all $g \in G$ and $s \in S$, by (1.49) there exist unique elements $t_s \in S$ and $k_s \in K$ such that $gs = t_s k_s$. Then we have

$$\sigma(g)f = \sum_{s\in S} \sigma(gs) f_{v_s} = \sum_{s\in S} \sigma(t_s)\sigma(k_s) f_{v_s} = \sum_{s\in S} \sigma(t_s) f_{\rho(k_s)v_s},$$

where in the last equality we used the fact that the map $v \mapsto f_v$ is an isomorphism of K-representations.

The following lemma is a converse to the above.

Lemma 1.6.2. *Let $K \le G$, and let S be a set of representatives of G/K. Let (τ, W) be a representation of G. Suppose that $V \le W$ is a K-invariant subspace and that*

$$W = \bigoplus_{s\in S} \tau(s)V$$

as vector spaces. Then $W \cong \operatorname{Ind}_K^G V$ as representations of G.

Proof. Define $\tilde{V}$ as in (1.51). For each $s \in S$, let $v_s = f(s)$. Then we have the isomorphism of vector spaces.

$$W = \bigoplus_{s\in S} \tau(s)V \to \bigoplus_{s\in S} \sigma(s)\tilde{V} = Z, \quad \sum_{s\in S} \tau(s)v_s \mapsto \sum_{s\in S} \sigma(s) f_{v_s}, \quad v_s \in V \ \forall\ s \in S. \tag{1.54}$$

For $g \in G$, we have

$$\begin{aligned}\tau(g)\left(\sum_{s\in S} \tau(s)v_s\right) &= \sum_{s\in S} \tau(gs)v_s = \sum_{s\in S} \tau(t_s)\tau(k_s)v_s \\ &\mapsto \sum_{s\in S} \sigma(t_s) f_{\tau(k_s)v_s} = \sum_{s\in S} \sigma(t_s)\sigma(k_s) f_{v_s} = \sigma(g)\sum_{s\in S}\sigma(s) f_{v_s}.\end{aligned}$$

Hence (1.54) is an isomorphism of G-representations. □

Recall that the *index* of the subgroup $K \le G$, is defined to be $[G : K] = |G/K|$. Since $|S| = |G/K|$, it follows immediately from Lemma 1.6.2 that

$$\dim\left(\operatorname{Ind}_K^G V\right) = [G : K] \dim V. \tag{1.55}$$

Proposition 1.6.3. *Suppose $K \leq G$ and let $X = G/K$. Then the permutation representation of G on $L(X)$ is isomorphic to $\operatorname{Ind}_K^G \iota$, where $(\iota, \mathbb{C})$ is the trivial representation of K.*

Proof. By definition,

$$\operatorname{Ind}_K^G \mathbb{C} = \{f \colon G \to \mathbb{C} : f(gk) = f(g), \text{ for all } g \in G, k \in K\}$$

is the space of right K-invariant functions on G. Thus the proposition follows from Theorem 1.5.11(a). ☐

Exercises.

1.6.1. (a) Prove that f_v, as defined in (1.50), is an element of Z.

(b) Prove that $\tilde{V}$, as defined in (1.51), is a K-invariant subspace of Z.

(c) Prove that the map $V \to \tilde{V}$, $v \mapsto f_v$, is an isomorphism of K-representations.

(d) Prove that equality (1.53) holds and that this is the unique way to write $f \in Z$ as a sum of elements of $\sigma(s)\tilde{V}$, $s \in S$. *Hint*: f is uniquely determined by its values on S.

1.6.2 ([CSST10, Ex. 1.6.3]). Suppose K is a subgroup of G and let S be a set of representatives of G/K. Let (π, W) be a representation of G, suppose that $V \leq W$ is K-invariant, and denote by (ρ, V) the corresponding K-representation. Prove that if $W = \langle \pi(s)V : s \in S \rangle$, then there exists a surjective intertwiner from $\operatorname{Ind}_K^G \rho$ to π.

Hint: If we let $\sigma = \operatorname{Ind}_K^G \rho$, then the required surjective intertwiner is the map

$$\sigma(s) f_v \mapsto \pi(s) v, \quad s \in S, \ v \in V,$$

extended by linearity.

1.6.2 First properties of induced representations

Induction is transitive in the following sense.

Proposition 1.6.4 (Transitivity of induction)**.** *Suppose $K \leq H \leq G$ are subgroups and let (ρ, V) be a representation of K. Then*

$$\operatorname{Ind}_H^G \left(\operatorname{Ind}_K^H V \right) \cong \operatorname{Ind}_K^G V$$

as representations of G.

Proof. Let $\sigma = \operatorname{Ind}_K^H \rho$. Consider the linear map

$$\operatorname{Ind}_H^G \left(\operatorname{Ind}_K^H V\right) = \operatorname{Fun}_H\left(G, \operatorname{Fun}_K(H,V)\right) \to \operatorname{Fun}(G,V) := \{f \colon G \to V\},$$
$$f \mapsto \tilde{f}, \quad \text{where } \tilde{f}(g) = f(g)(1_G),\ g \in G.$$

For $k \in K$, $g \in G$, and $f \in \operatorname{Hom}_H\left(G, \operatorname{Hom}_K(H,V)\right)$, we have

$$\begin{aligned}
\tilde{f}(gk) &= f(gk)(1_G) \\
&= \big(\sigma(k^{-1})f(g)\big)(1_G) && \text{(since } f \in \operatorname{Hom}_H(G, \operatorname{Hom}_K(H,V))) \\
&= f(g)(k) && \text{(by (1.48))} \\
&= \rho(k^{-1})\big(f(g)(1_G)\big) && \text{(since } f(g) \in \operatorname{Hom}_K(H,V)) \\
&= \rho(k^{-1})\tilde{f}(g)
\end{aligned}$$

Thus, $\tilde{f} \in \operatorname{Fun}_K(G,V)$.

Let $\eta = \operatorname{Ind}_H^G \sigma$ and $\pi = \operatorname{Ind}_K^G \rho$. Then, for $g_1, g_2 \in G$ and $f \in \operatorname{Hom}_H\left(G, \operatorname{Hom}_K(H,V)\right)$, we have

$$\begin{aligned}
\big(\pi(g_1)\tilde{f}\big)(g_2) &= \tilde{f}\left(g_1^{-1}g_2\right) \\
&= f\left(g_1^{-1}g_2\right)(1_G) \\
&= \big(\eta(g_1)f\big)(g_2)(1_G) \\
&= \big(\widetilde{\eta(g_1)f}\big)(g_2).
\end{aligned}$$

Thus the map $f \mapsto \tilde{f}$ is an intertwiner

$$\operatorname{Ind}_H^G \left(\operatorname{Ind}_K^H V\right) = \operatorname{Fun}_H\left(G, \operatorname{Hom}_K(H,V)\right) \to \operatorname{Fun}_K(G,V) = \operatorname{Ind}_K^G V.$$

Since

$$\begin{aligned}
\dim \operatorname{Ind}_H^G \left(\operatorname{Ind}_K^H V\right) = [G:H] \dim \operatorname{Ind}_K^H V = [G:H][H:K] \dim V \\
= [G:K] \dim V = \operatorname{Ind}_K^G V,
\end{aligned}$$

to show that $f \mapsto \tilde{f}$ is an isomorphism, it suffices to prove that it is injective. Suppose $\tilde{f} = 0$. Then, for all $g \in G$ and $h \in H$, we have

$$f(g)(h) = \big(\sigma(h^{-1})f(g)\big)(1_G) = f(gh)(1_G) = \tilde{f}(gh).$$

Thus $f = 0$, as desired. □

Theorem 1.6.5 (Frobenius character formula for induced representations)**.** *Let (ρ, W) be a representation of K, where $K \leq G$. Then*

$$\chi^{\operatorname{Ind}_K^G \rho}(g) = \sum_{s \in S : s^{-1}gs \in K} \chi^\rho\left(s^{-1}gs\right), \tag{1.56}$$

where S is any system of representatives for G/K.

Proof. A proof of this result can be found in [CSST10, Th. 1.6.7]. □

1.6.3 Frobenius reciprocity

The following fundamental result gives a precise relationship between the operations of induction and restriction.

Theorem 1.6.6 (Frobenius reciprocity). *Let G be a finite group, $K \le G$ a subgroup, (σ, W) a representation of G, and (ρ, V) a representation of K. Then we have an isomorphism of vector spaces*

$$\operatorname{Hom}_G\left(W, \operatorname{Ind}_K^G V\right) \to \operatorname{Hom}_K\left(\operatorname{Res}_K^G W, V\right), \quad T \mapsto \widehat{T}, \quad \textit{where}$$
$$\widehat{T} \colon W \to V, \quad \widehat{T}w = (Tw)(1_G).$$

Proof. We first check that $\widehat{T} \in \operatorname{Hom}_K\left(\operatorname{Res}_K^G W, V\right)$. Let $\tau = \operatorname{Ind}_K^G \rho$. For $k \in K$ and $w \in W$, we have

$$\begin{aligned}
\widehat{T}\sigma(k)w &= \big(T(\sigma(k)w)\big)(1_G) \\
&= \big(\tau(k)(Tw)\big)(1_G) && \left(T \in \operatorname{Hom}_G\left(\operatorname{Ind}_K^G V\right)\right) \\
&= (Tw)(k^{-1}) && \text{(by (1.48))} \\
&= \rho(k)\big(Tw(1_G)\big) && \text{(by (1.47))} \\
&= \rho(k)\widehat{T}w,
\end{aligned}$$

as desired.

Now consider the map

$$\operatorname{Hom}_K\left(\operatorname{Res}_K^G W, V\right) \to \operatorname{Hom}_G\left(W, \operatorname{Ind}_K^G V\right), \quad U \mapsto \check{U}, \quad \text{where}$$
$$\left(\check{U}w\right)(g) = U\sigma(g^{-1})w, \quad \text{for all } g \in G,\ w \in W.$$

It is straightforward to verify that $\check{U} \in \operatorname{Hom}_G\left(W, \operatorname{Ind}_K^G V\right)$ (Exercise 1.6.3). Let $T \in \operatorname{Hom}_G\left(W, \operatorname{Ind}_K^G V\right)$, and set $U = \widehat{T}$. Then

$$\left(\check{U}w\right)(g) = \widehat{T}\sigma(g^{-1})w = \left(T\sigma(g^{-1})w\right)(1_G) = \left(\tau(g^{-1})(T_w)\right)(1_G) = (Tw)(g).$$

Thus $\check{U} = T$. Similarly, let $U \in \operatorname{Hom}_K\left(\operatorname{Res}_K^G W, V\right)$ and set $T = \check{U}$. One can verify that $\widehat{T} = U$ (Exercise 1.6.4). □

Remark 1.6.7. In the language of category theory, Theorem 1.6.6 (together with a "naturality" statement) says that induction is *right adjoint* to restriction. Induction is also *left adjoint* to restriction, but this is an easier result that holds in a much greater level of generality. The fact that induction is right adjoint is an important property of what are called *Frobenius extensions*. The group algebras of nested finite groups are special cases of a Frobenius extensions.

Corollary 1.6.8. *Suppose, in the setting of Theorem 1.6.6 that W and V are irreducible. Then the multiplicity of W in $\operatorname{Ind}_K^G V$ is equal to the multiplicity of V in $\operatorname{Res}_K^G W$.*

Proof. This follows immediately from Theorem 1.6.6 and Lemma 1.2.5. □

Exercises.

1.6.3. Prove that $\check{U}$, as defined in the proof of Theorem 1.6.6 is an element of $\operatorname{Hom}_G\left(W, \operatorname{Ind}_K^G V\right)$.

1.6.4. In the notation of the proof of Theorem 1.6.6, let $U \in \operatorname{Hom}_K\left(\operatorname{Res}_K^G W, V\right)$ and set $T = \check{U}$. Prove that $\widehat{T} = U$.

1.6.4 Mackey's lemma and the intertwining number theorem

We now discuss a sort of "commutation" property for induction and restriction. Suppose H and K are two subgroups of G and let (ρ, W) be a representation of K. Let S be a system of representatives for the double cosets in $H\backslash G/K$, so that

$$G = \bigsqcup_{s\in S} HsK.$$

For each $s \in S$, let

$$G_s = sKs^{-1} \cap H \leq G,$$

and define a representation (ρ_s, W_s) of G_s by setting $W_s = W$ and

$$\rho_s(t)w = \rho(s^{-1}ts)w, \quad \text{for all } t \in G_s,\ w \in W_s.$$

Theorem 1.6.9 (Mackey's lemma)**.** *With notation as above, we have an isomorphism of H-representations*

$$\operatorname{Res}_H^G \operatorname{Ind}_K^G \rho \cong \bigoplus_{s\in S} \operatorname{Ind}_{G_s}^H \rho_s.$$

Proof. Let

$$Z_s = \left\{F\colon G \to W : F(hs'k) = \delta_{s,s'}\rho(k^{-1})F(hs)\ \forall\, h \in H,\ k \in K,\ s' \in S\right\}.$$

Comparing to (1.47), we see that Z_s is the subspace of $Z = \operatorname{Ind}_K^G W$ consisting of those functions that vanish outside HsK. Thus

$$Z = \bigoplus_{s\in S} Z_s \quad \text{as vector spaces.}$$

For each $s \in S$, the space Z_s is H-invariant (Exercise 1.6.6(a)). Therefore, it suffices to prove that, for all $s \in S$,

$$Z_s \cong \operatorname{Ind}_{G_s}^H W_s \quad \text{as representations of } H.$$

Consider the linear map

$$Z_s \to \operatorname{Fun}(H, W) = \{f\colon H \to W\}, \quad F \mapsto f_s,\ \text{where } f_s(h) = F(hs),\ \forall\ h \in H. \tag{1.57}$$

For $t \in G_s$, we have $s^{-1}ts \in K$, and so

$$f_s(ht) = F(hts) = F\left(hss^{-1}ts\right) = \rho\left(s^{-1}t^{-1}s\right)F(hs) = \rho_s\left(t^{-1}\right)f_s(h).$$

Thus $f_s \in \mathrm{Fun}_{G_s}(H, W_s) = \mathrm{Ind}_{G_s}^H W_s$. Hence (1.57) is a linear map

$$Z_s \to \mathrm{Ind}_{G_s}^H W_s.$$

We will now construct the inverse to (1.57). First consider the map

$$\begin{aligned} &\mathrm{Ind}_{G_s}^H W_s \to \mathrm{Fun}(G, W), \quad f \mapsto F_s \in \mathrm{Fun}(G, W), \text{ where} \\ &F_s(hs'k) = \delta_{s,s'}\rho(k^{-1})f(h), \quad \text{for all } k \in K,\ h \in H,\ s' \in S. \end{aligned} \tag{1.58}$$

We must verify that F_s is well defined. If $hsk = h_1sk_1$ for some $h, h_1 \in H$ and $k, k_1 \in K$, then $t := skk_1^{-1}s^{-1} = h^{-1}h_1 \in G_s$, and so

$$\begin{aligned} \rho\left(k_1^{-1}\right)f(h_1) &= \rho\left(k^{-1}\right)\left(\rho\left(s^{-1}h^{-1}h_1s\right)f(h_1)\right) \qquad (\text{since } k_1^{-1} = k^{-1}s^{-1}h^{-1}h_1s) \\ &= \rho\left(k^{-1}\right)\left(\rho_s(t)f(h_1)\right) \\ &= \rho\left(k^{-1}\right)f\left(h_1t^{-1}\right) \\ &= \rho\left(k^{-1}\right)f(h). \end{aligned}$$

Thus F_s is well defined. For $h \in H$, $k \in K$, and $s' \in S$, we have

$$F_s(hs'k) = \delta_{s,s'}\rho(k^{-1})f(h) = \delta_{s,s'}\rho(k^{-1})F_s(hs),$$

so $F_s \in Z_s$. Hence (1.58) is a linear map

$$\mathrm{Ind}_{G_s}^H W_s \to Z_s.$$

It is straightforward to verify that the maps (1.57) and (1.58) are mutually inverse and that (1.57) intertwines the H-action (Exercise 1.6.6(b)). This completes the proof. □

Let us now consider the special case where (ρ, W) is the permutation representation on $L(X)$, with $X = G/K$. Let $x_0 = K \in G/K$, so that K is the stabilizer of x_0. As above, we let S be a set of representatives for the double cosets $H\backslash G/K$. By Exercise 1.6.5, G_s is the stabilizer of sx_0 for each $s \in S$. Then

$$X = \bigsqcup_{s \in S} \Omega_s, \quad \Omega_s = Hsx_0 = \{hsx_0 : h \in H\},$$

is the decomposition of X into H-orbits.

Now let $\rho = \iota_K$ be the trivial representation of K. Then ρ_s is the trivial representation of G_s, and $\mathrm{Ind}_{G_s}^H \rho_s$ is the permutation representation of H on $L(\Omega_s)$ by Proposition 1.6.3.

Thus, Mackey's lemma gives the decomposition

$$\mathrm{Res}_H^G L(X) = \bigoplus_{s \in S} L\left(\Omega_s\right). \tag{1.59}$$

We finish this chapter with an important application of Mackey's lemma.

Theorem 1.6.10 (Intertwining number theorem)**.** *With the same hypotheses as in Mackey's lemma (Theorem 1.6.9), assume that σ is a representation of H. Then*

$$\dim \operatorname{Hom}_G \left(\operatorname{Ind}_H^G \sigma, \operatorname{Ind}_K^G \rho\right) = \sum_{s\in S} \dim \operatorname{Hom}_{G_s} \left(\operatorname{Res}_{G_s}^H \sigma, \rho_s\right).$$

Proof. We have

$$\begin{aligned}
&\dim \operatorname{Hom}_G \left(\operatorname{Ind}_H^G \sigma, \operatorname{Ind}_K^G \rho\right) \\
&\quad = \dim \operatorname{Hom}_H \left(\sigma, \operatorname{Res}_H^G \operatorname{Ind}_K^G \rho\right) && \text{(Frobenius reciprocity (Theorem 1.6.6))} \\
&\quad = \sum_{s\in S} \dim \operatorname{Hom}_H \left(\sigma, \operatorname{Ind}_{G_s}^H \rho_s\right) && \text{(Mackey's lemma (Theorem 1.6.9))} \\
&\quad = \sum_{s\in S} \dim \operatorname{Hom}_{G_s} \left(\operatorname{Res}_{G_s}^H \sigma, \rho_s\right) && \text{(Frobenius reciprocity (Theorem 1.6.6)).}
\end{aligned}$$

□

Exercises.

1.6.5 ([CSST10, Ex. 1.6.13]). Identify $H\backslash G/H$ with the set of H-orbits on $X = G/K$. Prove that G_s is the stabilizer in H of the point $x_s = sK$. (Compare with Lemma 1.5.9.)

1.6.6. (a) Prove that Z_s is H-invariant.

(b) Verify that (1.57) and (1.58) are mutually inverse, and that (1.57) intertwines the H-action.

Chapter 2

The theory of Gelfand–Tsetlin bases

Our goal in this chapter is to develop the theory of Gelfand–Tsetlin bases for group algebras and permutation representations. These are bases that are well suited to the restriction of a representation to certain chains of subgroups. We closely follow the presentation in [CSST10, Ch. 2]. Throughout this chapter we again suppose that G is a finite group.

2.1 Algebras of conjugacy invariant functions

2.1.1 Conjugacy invariant functions

Suppose H is a subgroup of G. A function $f \in L(G)$ is H-*conjugacy invariant* if

$$f\left(h^{-1}gh\right) = f(g), \quad \text{for all } h \in H,\ g \in G.$$

We let $\mathcal{C}(G,H)$ denote the set of all H-conjugacy invariant functions on G. This is a subalgebra of $L(G)$ (under convolution). Indeed, for $f_1, f_2 \in \mathcal{C}(G,H)$, we have

$$\begin{aligned}
(f_1 * f_2)\left(h^{-1}gh\right) &= \sum_{s\in G} f_1\left(h^{-1}ghs\right) f_2\left(s^{-1}\right) \\
&= \sum_{s\in G} f_1\left(ghsh^{-1}\right) f_2\left(hs^{-1}h^{-1}\right) && (\text{since } f_1, f_2 \in \mathcal{C}(G,H)) \\
&= \sum_{t\in G} f_1(gt) f_2\left(t^{-1}\right) && \left(t = hsh^{-1}\right) \\
&= (f_1 * f_2)(g).
\end{aligned}$$

Hence $f_1 * f_2 \in \mathcal{C}(G,H)$. Note that $\mathcal{C}(G,G)$ is the algebra of central functions on G (Definition 1.3.1).

Consider the action of $G \times H$ on G defined by

$$(g,h)\cdot g_0 = gg_0h^{-1}, \quad \text{for all } g, g_0 \in G,\ h \in H. \tag{2.1}$$

We denote the associated permutation representation by η, so that

$$\big(\eta(g,h)f\big)(g_0) = f\left(g^{-1}g_0h\right), \quad \text{for all } f \in L(G),\ g, g_0 \in G,\ h \in H. \tag{2.2}$$

Lemma 2.1.1. *(a) The stabilizer of 1_G under the action (2.1) is*

$$\tilde{H} = \{(h, h) : h \in H\} \leq G \times H.$$

(b) Let $L(\tilde{H} \backslash (G \times H)/\tilde{H})$ denote the algebra of bi-$\tilde{H}$-invariant functions on $G \times H$. Then we have an isomorphism of algebras

$$\Phi \colon L(\tilde{H} \backslash (G \times H)/\tilde{H}) \to \mathcal{C}(G, H), \quad \Phi(F)(g) = |H|\, F(g, 1_G).$$

Proof. Part (a) is clear since

$$1_G = (g, h) \cdot 1_G = gh^{-1} \iff g = h.$$

We now prove part (b).

Suppose $F \in L(\tilde{H} \backslash (G \times H)/\tilde{H})$ and let $f = \Phi(F)$. Then

$$\begin{aligned} f\left(h^{-1}gh\right) &= |H|\, F\left(h^{-1}gh, h^{-1}h\right) \\ &= |H|\, F\left(g, 1_G\right) && \text{(since } F \text{ is bi-}\tilde{H}\text{-invariant)} \\ &= f(g). \end{aligned}$$

Thus $f \in \mathcal{C}(G, H)$. It is clear that Φ is linear. To see that Φ is injective, note that

$$\begin{aligned} F(g, h) &= F\left(gh^{-1}, 1_G\right) && \text{(since } F \text{ is right } \tilde{H}\text{-invariant)} \\ &= \frac{1}{|H|} f\left(gh^{-1}\right) && \text{(where } f = \Phi(F)\text{)}. \end{aligned}$$

Thus F is uniquely determined by f. To prove Φ is surjective, let $f \in \mathcal{C}(G, H)$. Then, if we define $F(g, h) = \frac{1}{|H|} f\left(gh^{-1}\right)$, we have that F is bi-$\tilde{H}$-invariant, and $f = \Phi(F)$.

It remains to prove that Φ is multiplicative. Let $F_1, F_2 \in L(\tilde{H} \backslash (G \times H)/\tilde{H})$. Then, for all $g \in G$, we have

$$\begin{aligned} \big(\Phi(F_1 * F_2)\big)(g) &= |H|\, (F_1 * F_2)(g, 1_G) \\ &= |H| \sum_{s \in G} \sum_{h \in H} F_1(gs, h) F_2\left(s^{-1}, h^{-1}\right) \\ &= |H| \sum_{s \in G} \sum_{h \in H} F_1(gsh^{-1}, 1_G) F_2\left(hs^{-1}, 1_G\right) && (F_1, F_2 \text{ are bi-}\tilde{H}\text{-invariant}) \\ &= |H|^2 \sum_{t \in G} F_1(gt, 1_G) F_2\left(t^{-1}, 1_G\right) && \left(sh^{-1} = t\right) \\ &= \sum_{t \in G} \left(\Phi(F_1)(gt)\right) \left(\Phi(F_2)(t^{-1})\right) \\ &= \big(\Phi(F_1) * \Phi(F_2)\big)(g). \end{aligned}$$

□

We now wish to decompose the permutation representation η of (2.2) into irreducible $G \times H$-subrepresentations. Recall that, by Theorem 1.3.18, every irreducible representation of $G \times H$ is of the form $\sigma \boxtimes \rho$ for some $\sigma \in \widehat{G}$ and $\rho \in \widehat{H}$. Note also that the adjoint of $\operatorname{Res}^G_H \sigma$ is $\operatorname{Res}^G_H \sigma'$, where σ' is the adjoint of σ (Exercise 2.1.1).

Theorem 2.1.2. *Suppose* $(\sigma, V) \in \widehat{G}$ *and* $(\rho, W) \in \widehat{H}$. *Let* $(\rho', W') \in \widehat{H}$ *denote the adjoint of* (ρ, W). *Then we have an isomorphism of vector spaces*

$$\operatorname{Hom}_{G\times H}(\sigma \boxtimes \rho, \eta) \to \operatorname{Hom}_H\left(\rho, \operatorname{Res}^G_H \sigma'\right), \quad T \mapsto \tilde{T}, \ \textit{where}$$
$$\big(\tilde{T}w\big)(v) = \big(T(v \otimes w)\big)(1_G), \quad \textit{for all } v \in V, \ w \in W.$$

Proof. Note that a linear map $T\colon V \otimes W \to L(G)$ lies in $\operatorname{Hom}_{G\times H}(\sigma \boxtimes \rho, \eta)$ if and only if

$$\big(T(\sigma(g)v \otimes \rho(h)w)\big)(g_0) = \big(T(v \otimes w)\big)\left(g^{-1}g_0h\right), \ \forall\, g, g_0 \in G, \ h \in H, \ v \in V, \ w \in W. \tag{2.3}$$

Let $T \in \operatorname{Hom}_{G\times H}(\sigma \boxtimes \rho, \eta)$. For $h \in H$, $v \in V$, and $w \in W$, we have

$$\begin{aligned}
\left(\tilde{T}\rho(h)w\right)(v) &= \big(T(v \otimes \rho(h)w)\big)(1_G) \\
&= \big(T(v \otimes w)\big)(h) && \text{(by (2.3))} \\
&= \big(T(\sigma(h^{-1})v \otimes w)\big)(1_G) && \text{(by (2.3))} \\
&= \left(\tilde{T}w\right)\left(\sigma\left(h^{-1}\right)v\right) \\
&= \left(\sigma'(h)\tilde{T}w\right)(v) && \text{(by (1.9))}.
\end{aligned}$$

Thus

$$\sigma'(h)\tilde{T} = \tilde{T}\rho(h), \quad \text{for all } h \in H,$$

and so $\tilde{T} \in \operatorname{Hom}_H\left(\rho, \operatorname{Res}^G_H \sigma'\right)$, as desired.

To see that the map $T \mapsto \tilde{T}$ is injective, note that, by (2.3), we have

$$\big(T(v \otimes w)\big)(g) = \left(T\left(\sigma\left(g^{-1}\right)v \otimes w\right)\right)(1_G) = \left(\tilde{T}w\right)\left(\sigma\left(g^{-1}\right)v\right). \tag{2.4}$$

Thus T is uniquely determined by $\tilde{T}$.

Since the map $T \mapsto \tilde{T}$ is clearly linear, it remains to show it is surjective. Suppose $S \in \operatorname{Hom}_H\left(\rho, \operatorname{Res}^G_H \sigma'\right)$. Define $T \in \operatorname{Hom}(V \otimes W, L(G))$ by

$$\big(T(v \otimes w)\big)(g_0) = (Sw)\left(\sigma\left(g_0^{-1}\right)v\right), \quad \text{for all } g_0 \in G, \ v \in V, \ w \in W. \tag{2.5}$$

Then, for all $g \in G$, $h \in H$, $v \in V$, and $w \in W$, we have

$$\begin{aligned}
\big(T(\sigma(g)v \otimes \rho(h)w)\big)(g_0) &= (S\rho(h)w)\left(\sigma\left(g_0^{-1}g\right)v\right) \\
&= (\sigma'(h)Sw)\left(\sigma\left(g_0^{-1}g\right)v\right) && \left(S \in \operatorname{Hom}_H\left(\rho, \operatorname{Res}^G_H \sigma'\right)\right) \\
&= (Sw)\left(\sigma\left(h^{-1}g_0^{-1}g\right)v\right) && \text{(by (1.9))} \\
&= (T(v \otimes w))\left(g^{-1}g_0h\right).
\end{aligned}$$

Hence T satisfies (2.3), and so $T \in \operatorname{Hom}_{G\times H}(\sigma \boxtimes \rho, \eta)$. Comparing (2.4) and (2.5), we see that $\tilde{T} = S$. □

Corollary 2.1.3. *The multiplicity of* $\sigma \boxtimes \rho$ *in* η *is equal to the multiplicity of* ρ *in* $\operatorname{Res}^G_H \sigma'$.

Proof. This follows from Lemma 1.2.5 and Theorem 2.1.2. □

For $\sigma \in \widehat{G}$ and $\rho \in \widehat{H}$, let

$$m_{\rho,\sigma} = \dim \operatorname{Hom}_H \left(\rho, \operatorname{Res}^G_H \sigma'\right)$$

denote the multiplicity of ρ in $\operatorname{Res}^G_H \sigma'$.

Corollary 2.1.4. *The decomposition of η into irreducible subrepresentations of $G \times H$ is given by*

$$\eta \cong \bigoplus_{\sigma\in\widehat{G}} \bigoplus_{\rho\in\widehat{H}} (\sigma \boxtimes \rho)^{\oplus m_{\rho,\sigma}}.$$

We now consider the case where $H = G$, so that $(\eta, L(G))$ is a representation of $G \times G$. We showed in Example 1.5.14 that $(G \times G, \tilde{G})$ is a Gelfand pair. This also follows from the following result.

Corollary 2.1.5. *Suppose that $H = G$. We have a decomposition into $G \times G$-invariant subspaces*

$$L(G) = \bigoplus_{\rho\in\widehat{G}} M_\rho,$$

where M_ρ is the subspace of $L(G)$ spanned by all matrix coefficients

$$\varphi(g) = \langle \rho(g)v, w\rangle_{W_\rho}, \quad v, w \in W_\rho.$$

Furthermore, the action of $G \times G$ on the summand M_ρ is isomorphic to $\rho' \boxtimes \rho$. Hence we have an isomorphism of $G \times G$-representations

$$\eta \cong \bigoplus_{\rho\in\widehat{G}} \rho' \boxtimes \rho.$$

Proof. We have

$$m_{\rho,\sigma} = \dim \operatorname{Hom}_G(\rho, \sigma') = \begin{cases} 1 & \text{if } \rho \sim \sigma', \\ 0 & \text{otherwise.} \end{cases}$$

Define $T \in \operatorname{Hom}_{G\times G}(\rho' \boxtimes \rho, \eta)$ as in (2.5) with $\sigma = \rho'$ and $S = I_{W_\rho}$. Then, for all $v, w \in W_\rho$ and $g \in G$, we have

$$\begin{aligned} \big(T(\theta_v \otimes w)\big)(g) &= \left(\rho'\left(g^{-1}\right)\theta_v\right)(w) \\ &= \theta_v(\rho(g)w) \\ &= \langle \rho(g)w, v\rangle, \end{aligned}$$

where, in the first equality, we use the natural identification of $(\rho')'$ with ρ (i.e. of the double dual of a vector space with the vector space itself.) Hence $T(\theta_v \otimes w) \in M_\rho$.

Since $\rho' \boxtimes \rho$ is irreducible and $\dim M_\rho = d_\rho^2$ (a basis for M_ρ is given by the matrix coefficients $\varphi^\rho_{i,j}$, $1 \le i, j \le d_\rho$, by Corollary 1.5.6), it follows that $T(W'_\rho \otimes W_\rho) = M_\rho$, completing the proof. □

Exercises.

2.1.1. Prove that if σ is a representation of G with adjoint σ', then the adjoint of $\operatorname{Res}^G_H \sigma$ is $\operatorname{Res}^G_H \sigma'$.

2.1.2. (a) Using the properties of the matrix coefficients, prove directly that $M_\rho \cong W'_\rho \otimes W_\rho$, in the language of the proof of Corollary 2.1.5. Then deduce Corollary 2.1.5.

(b) Use part (a) to prove Corollary 2.1.4.

2.1.2 Multiplicity-free subgroups

Definition 2.1.6 (Multiplicity-free subgroup)**.** A subgroup H of G is said to be *multiplicity free* if, for every $\sigma \in \widehat{G}$, the restriction $\operatorname{Res}^G_H \sigma$ is multiplicity free or, equivalently, if

$$\dim \operatorname{Hom}_H \left(\rho, \operatorname{Res}^G_H \sigma\right) \le 1, \quad \text{for all } \rho \in \widehat{H},\ \sigma \in \widehat{G}.$$

Theorem 2.1.7. *The following conditions are equivalent:*

(a) The algebra $\mathcal{C}(G, H)$ is commutative.

(b) $(G \times H, \tilde{H})$ is a Gelfand pair. (Recall that $\tilde{H} = \{(h, h) : h \in H\}$.)

(c) H is a multiplicity-free subgroup of G.

Proof. The equivalence of (b) and (c) follows from Corollary 2.1.4, Lemma 2.1.1(a), and the definition of a Gelfand pair (Definition 1.4.7). The equivalence of (a) and (b) follows from Lemma 2.1.1(b) and Corollary 1.5.12. □

Remark 2.1.8. (a) When $H = G$, the conditions of Theorem 2.1.7 are always satisfied:

- $\mathcal{C}(G, G)$ is the space of central functions, which is the centre of the group algebra $L(G)$ (see Remark 1.5.3). Thus it is commutative.
- $(G \times G, \tilde{G})$ is a Gelfand pair by Example 1.5.14.
- Clearly G is a multiplicity-free subgroup of itself.

(b) When $H = \{1_G\}$, the conditions in Theorem 2.1.7 are equivalent to G being abelian since $\mathcal{C}(G, \{1_G\})$ is commutative if and only if G is commutative. (This follows, for example, from (1.39).)

Proposition 2.1.9. *We have that $(G \times H, \tilde{H})$ is a symmetric Gelfand pair if and only if*

$$\forall\, g \in G\ \exists\, h \in H \text{ such that } hgh^{-1} = g^{-1}$$

(that is, every element of G is H-conjugate to its inverse). Moreover, if this is the case, then H is a multiplicity free subgroup of G.

Proof. By Exercise 1.5.10, the pair $(G \times H, \tilde{H})$ is symmetric if and only if for all $(g,h) \in G \times H$, there exist $h_1, h_2 \in H$ such that

$$\left(g^{-1}, h^{-1}\right) = (g,h)^{-1} = (h_1,h_1)(g,h)(h_2,h_2) = (h_1gh_2, h_1hh_2). \tag{2.6}$$

Taking $h = 1_G$, we obtain $h_2 = h_1^{-1}$ and so $g^{-1} = h_1gh_1^{-1}$.

To prove the converse implication, suppose that every element of G is H-conjugate to its inverse. Then, for $(g,h) \in G \times H$, we can choose $t \in H$ such that

$$\left(gh^{-1}\right)^{-1} = t\left(gh^{-1}\right)t^{-1}.$$

Thus, taking $h_1 = h^{-1}t$ and $h_2 = h^{-1}t^{-1}$, we see that

$$(h_1gh_2, h_1hh_2) = \left(h^{-1}tgh^{-1}t^{-1}, h^{-1}thh^{-1}t^{-1}\right) = \left(h^{-1}hg^{-1}, h^{-1}\right) = \left(g^{-1}, h^{-1}\right).$$

Hence (2.6) is satisfied. □

Exercises.

2.1.3. Suppose G is abelian. Show that $(G \times H, \tilde{H})$ is a symmetric Gelfand pair if and only if every $g \in G$ satisfies $g^2 = 1_G$.

2.2 Gelfand–Tsetlin bases

2.2.1 Branching graphs and Gelfand–Tsetlin bases

A chain

$$\{1_G\} = G_1 \le G_2 \le \cdots \le G_{n-1} \le G_n = G \tag{2.7}$$

of subgroups is said to be *multiplicity free* if G_{k-1} is a multiplicity-free subgroup of G_k for $1 < k \le n$. Note that, by Remark 2.1.8((b)), if (2.7) is multiplicity free, then G_2 is abelian.

From now on, we fix a multiplicity-free chain (2.7). The *branching graph* of this chain is the directed graph with vertex set

$$\bigsqcup_{k=1}^{n} \widehat{G_k}$$

and edge set

$$\left\{(\rho,\sigma) \in \widehat{G_k} \times \widehat{G_{k-1}} : \sigma \text{ is a subrepresentation of } \operatorname{Res}^{G_k}_{G_{k-1}} \rho,\ 2 \le k \le n\right\}.$$

We will write $\rho \to \sigma$ if (ρ,σ) is an edge of the branching graph.

Suppose $(\rho, V_\rho) \in \widehat{G_n}$. Then

$$\operatorname{Res}^{G_n}_{G_{n-1}} V_\rho = \bigoplus_{\sigma \in \widehat{G_{n-1}} : \rho \to \sigma} V_\sigma$$

is an orthogonal decomposition. Then, for each $\sigma \in \widehat{G_{n-1}}$ we have an orthogonal decomposition

$$\operatorname{Res}^{G_{n-1}}_{G_{n-2}} V_\sigma = \bigoplus_{\theta \in \widehat{G_{n-2}} : \sigma \to \theta} V_\theta.$$

We continue in this way until, after the restriction from G_2 to G_1, we are left with sums of one-dimensional trivial representations.

To keep track of the restrictions, let

$$\mathcal{T}(\rho) = \{T : T = (\rho = \rho_n \to \rho_{n-1} \to \rho_{n-2} \to \cdots \to \rho_2 \to \rho_1),\ \rho_k \in \widehat{G_k},\ 1 \le k \le n-1.\}$$

denote the set of all paths T in the branching graph. (We will always be interested in *directed* paths.) Then we have

$$V_\rho = \bigoplus_{\substack{\rho_{n-1}: \\ \rho \to \rho_{n-1}}} V_{\rho_{n-1}} = \bigoplus_{\substack{\rho_{n-1}: \\ \rho \to \rho_{n-1}}} \bigoplus_{\substack{\rho_{n-2}: \\ \rho_{n-1} \to \rho_{n-2}}} V_{\rho_{n-2}} = \cdots = \bigoplus_{T \in \mathcal{T}(\rho)} V_{\rho_1}. \tag{2.8}$$

Since ρ_1 is the trivial representation of $G_1 = \{1_G\}$, it is one-dimensional. Therefore, for each $T \in \mathcal{T}(\rho)$, we can choose $v_T \in V_{\rho_1}$ with $\|v_T\| := \sqrt{\langle v_T, v_T \rangle} = 1$. (Thus v_T is defined up to a scalar factor of norm one.) Then (2.8) becomes

$$V_\rho = \bigoplus_{T \in \mathcal{T}(\rho)} \mathbb{C} v_T. \tag{2.9}$$

In other words,

$$\{v_T : T \in \mathcal{T}(\rho)\}$$

is an orthonormal basis of V, called a *Gelfand–Tsetlin basis* for V_ρ with respect to the multiplicity-free chain (2.7).

The branching graph and Gelfand–Tsetlin bases are well adapted to restriction to the subgroups appearing in the corresponding multiplicity free chain. In particular, if $\theta \in \widehat{G_k}$ for some $1 \le k \le n-1$, then the multiplicity of θ in $\operatorname{Res}^{G_n}_{G_k} \rho$ is equal to the number of paths from ρ to θ in the branching graph. Furthermore, we obtain an orthogonal decomposition of the θ-isotypic component of $\operatorname{Res}^{G_n}_{G_k} \rho$ into irreducible G_k-subrepresentations (each isomorphic to V_θ). Namely, we have a unique component V_θ for V_ρ for each path from ρ to θ, and the components corresponding to distinct paths are orthogonal.

For $j = 1, 2, \dots, n$ and $\rho \in \widehat{G}$, we let

$$\mathcal{T}_j(\rho) = \{S : S = (\rho = \rho_n \to \rho_{n-1} \to \cdots \to \rho_{j+1} \to \rho_j),\ \rho_k \in \widehat{G_k},\ j \le k \le n-1\}.$$

In particular, $\mathcal{T}_1(\rho) = \mathcal{T}(\rho)$. For

$$T = (\rho = \rho_n \to \rho_{n-1} \to \cdots \to \rho_2 \to \rho_1) \in \mathcal{T}(\rho),$$

we define the j-th *truncation* of T to be

$$T_j = (\rho = \rho_n \to \rho_{n-1} \to \cdots \to \rho_{j+1} \to \rho_j) \in \mathcal{T}_j(\rho).$$

For $1 \le j \le n$ and $S \in \mathcal{T}_j(\rho)$, we define

$$V_S = \bigoplus_{\substack{T \in \mathcal{T}(\rho): \\ T_j = S}} V_{\rho_1}, \qquad \rho_S = \left(\operatorname{Res}^G_{G_j} \rho\right)\Big|_{V_S}. \tag{2.10}$$

Then (ρ_S, V_S) is an irreducible G_j-representation, and $\rho_S \sim \rho_j$.

Exercises.

2.2.1. Fix a positive integer n and let G_n be the cyclic group of order 2^n with generator a. For $0 \le k \le n$, let G_k be the subgroup of G_n generated by $a^{2^{n-k}}$.

(a) Prove that $\{1_G\} = G_0 \le G_1 \le \cdots \le G_n$ is a multiplicity-free chain of subgroups.

(b) Describe the branching graph for this chain.

(c) Draw the branching graph for $n = 3$.

2.2.2 Gelfand–Tsetlin algebras

Let H be a subgroup of G. We can extend any $f \in L(H)$ by zero to a function $f^G_H \in L(G)$ defined by

$$f^G_H(g) = \begin{cases} f(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

If $H \le K \le G$, it is straightforward to verify that, for all $f, f_1, f_2 \in L(H)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$, we have

- $\left(f^K_H\right)^G_K = f^G_H$,
- $(f_1 * f_2)^G_H = (f_1)^G_H * (f_2)^G_H$ (Exercise 2.2.2), and
- $(\alpha_1 f_1 + \alpha_2 f_2)^G_H = \alpha_1 (f_1)^G_H + \alpha_2 (f_2)^G_H$.

Thus we can view $L(H)$ as a subalgebra of $L(G)$. By an abuse of notation, we will often denote the extension f^G_H again by f.

Recalling Definition 1.3.12, for all $\rho \in \widehat{G}$ and $f \in L(H)$, we have

$$\rho\left(f^G_H\right) = \sum_{g \in G} f^G_H(g)\rho(g) = \sum_{h \in H} f(h)\rho(h) = \sum_{h \in H} f(h)\left(\operatorname{Res}^G_H \rho\right)(h) = \left(\operatorname{Res}^G_H \rho\right)(f).$$

Definition 2.2.1 (Gelfand–Tsetlin algebra)**.** For $1 \le k \le n$, we let $Z(k)$ denote the center of the group algebra $L(G_k)$, that is, $Z(k)$ is the subalgebra of central functions on G_k (see Remark 1.5.3). The *Gelfand–Tsetlin algebra* $\mathrm{GZ}(n)$ associated with the multiplicity-free chain (2.7) is the subalgebra of $L(G_n)$ generated by the subalgebras

$$Z(1), Z(2), \ldots, Z(n).$$

Theorem 2.2.2. *The Gelfand–Tsetlin algebra* $\mathrm{GZ}(n)$ *is a maximal commutative subalgebra of* $L(G_n)$*. Furthermore*

$$\mathrm{GZ}(n) = \{f \in L(G) : \rho(f)v_T \in \mathbb{C}v_T, \textit{ for all } \rho \in \widehat{G_n} \textit{ and } T \in \mathcal{T}(\rho)\}. \tag{2.11}$$

In other words, $\mathrm{GZ}(n)$ *is the subalgebra of* $L(G_n)$ *consisting of those* $f \in L(G_n)$ *whose Fourier transforms* $\rho(f)$*,* $\rho \in \widehat{G_n}$ *act diagonally on the Gelfand–Tsetlin basis of* V_ρ*.*

Proof. For $f_i \in Z(i)$ and $f_j \in Z(j)$ with $i \le j$, we have $f_i \in L(G_i) \le L(G_j)$, and so $f_i * f_j$, since $f_j \in Z(j)$. Thus $\mathrm{GZ}(n)$ is commutative and spanned by the products

$$f_1 * f_2 * \cdots * f_n, \quad f_k \in Z(k),\ 1 \le k \le n.$$

Let $\mathcal{A}$ denote the right-hand side of (2.11). Then $\mathcal{A}$ is an algebra by the multiplicative property of the Fourier transform (see (1.42)).

Suppose $1 \le j \le n$. Let $f_j \in Z(j)$, $\rho \in \widehat{G_n}$, and $T \in \mathcal{T}(\rho)$. Let $S = T_j$. Then

$$\begin{aligned} \rho(f_j)v_T &= \sum_{g\in G} f_j(g)\rho(g)v_T \\ &= \sum_{g\in G_j} f_j(g)\left(\operatorname{Res}^G_{G_j}\rho\right)(g)v_T && \text{(since } f_j \in L(G_j)) \\ &= \sum_{g\in G_j} f_j(g)\rho_S(g)v_T && \text{(by (2.10), since } v_T \in V_S) \\ &= \rho_S(f_j)v_T \\ &\in \mathbb{C}v_T && \text{(Lemma 1.3.13).} \end{aligned} \tag{2.12}$$

Thus $Z(j) \subseteq \mathcal{A}$ for each $1 \le j \le n$. Hence $\mathrm{GZ}(n) \subseteq \mathcal{A}$.

We now prove that $\mathcal{A} \subseteq \mathrm{GZ}(n)$. Let $\rho \in \widehat{G_n}$ and

$$T = (\rho = \rho_n \to \rho_{n-1} \to \cdots \to \rho_1) \in \mathcal{T}(\rho).$$

By Theorem 1.5.7, we can choose $f_j \in L(G_j)$, $1 \le j \le n$, such that, for all $\sigma \in \widehat{G_j}$, we have

$$\sigma(f_j) = \begin{cases} I_{V_{\rho_j}} & \text{if } \sigma = \rho_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$F_T = f_1 * f_2 * \cdots * f_n.$$

As in (2.12), for $S \in \mathcal{T}(\rho)$, we have

$$\rho(F_T)v_S = \begin{cases} v_T & \text{if } S = T, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Therefore, $\{F_T : T \in \mathcal{T}(\rho),\ \rho \in \widehat{G_n}\}$ is a basis for $\mathcal{A}$, and so $\mathcal{A} \subseteq \mathrm{GZ}(n)$.

By Theorem 1.5.7, we have

$$L(G_n) \cong \bigoplus_{\rho \in \widehat{G_n}} \mathrm{Hom}(V_\rho, V_\rho) \cong \bigoplus_{\rho \in \widehat{G_n}} M_{d_\rho, d_\rho}(\mathbb{C}).$$

Thus $\mathcal{A}$ is a maximal commutative subalgebra of $L(G_n)$ by Exercise 1.5.6. □

Corollary 2.2.3. *Every element of the Gelfand–Tsetlin basis of V_ρ is a common eigenvector for all $\rho(f)$, $f \in \mathrm{GZ}(n)$. In particular, it is uniquely determined, up to a scalar factor, by the corresponding eigenvalues.*

Proof. The last statement follows from (2.13). □

Suppose $f_1, f_2, \dots, f_n \in \mathrm{GZ}(n)$. By Corollary 2.2.3, for all $\rho \in \widehat{G_n}$, $T \in \mathcal{T}(\rho)$, and $1 \leq i \leq n$, we have

$$\rho(f_i)v_T^\rho = \alpha_{\rho,T,i} v_T^\rho \text{ for some } \alpha_{\rho,T,i} \in \mathbb{C}, \tag{2.14}$$

where v_T^ρ is the GZ-vector associated with the path T.

When the map

$$\widehat{G_n} \times \mathcal{T}(\rho) \to \mathbb{C}^n, \quad (\rho, T) \mapsto (\alpha_{\rho,T,1}, \alpha_{\rho,T,2}, \dots, \alpha_{\rho,T,n}),$$

is injective, we say that $f_1, \dots, f_n$ *separate* the vectors of the GZ-bases $\{v_T^\rho : \rho \in \widehat{G_n},\ T \in \mathcal{T}(\rho)\}$.

Proposition 2.2.4. *Let $G_1 \leq G_2 \leq \cdots \leq G_{n-1} \leq G_n$ be a multiplicity-free chain of groups. Then $\mathcal{C}(G_n, G_{n-1}) \subseteq \mathrm{GZ}(n)$.*

Proof. By Theorem 2.2.2, it suffices to prove that, for all $f \in \mathcal{C}(G_n, G_{n-1})$, the elements v_T, $T \in \mathcal{T}(\rho)$, of the Gelfand-Tsetlin basis are eigenvectors for $\rho(f)$, for all $\rho \in \widehat{G_n}$.

Note that

$$\begin{aligned} f \in \mathcal{C}(G_n, G_{n-1}) &\iff f(h^{-1}gh) = f(g), \quad \text{for all } h \in G_{n-1},\ g \in G_n \\ &\iff f(gh) = f(hg), \quad \text{for all } h \in G_{n-1},\ g \in G_n \\ &\iff (f * \delta_h)(g) = (\delta_h * f)(g), \quad \text{for all } h \in G_{n-1}, g \in G_n \\ &\iff f * \delta_h = \delta_h * f, \quad \text{for all } h \in G_{n-1}. \end{aligned}$$

Let $f \in \mathcal{C}(G_n, G_{n-1})$. Then, for all $\rho \in \widehat{G_n}$ and $h \in G_{n-1}$, we have

$$\begin{aligned} \rho(h)\rho(f) &= \rho(\delta_h * f) \\ &= \rho(f * \delta_h) \end{aligned}$$

$$= \rho(f)\rho(h).$$

Thus

$$\rho(f) \in \operatorname{End}_{G_{n-1}}\left(\operatorname{Res}^{G_n}_{G_{n-1}} \rho\right).$$

Since $\operatorname{Res}^{G_n}_{G_{n-1}}$ is multiplicity free, it then follows from Schur's lemma that

$$\rho(f)V_\sigma \subseteq V_\sigma, \quad \text{for all } f \in \mathcal{C}(G_n, G_{n-1}),\ \sigma \in \widehat{G_{n-1}},\ V_\sigma \le V_\rho.$$

Now, note that

$$\mathcal{C}(G_n, G_{n-1}) \subseteq \mathcal{C}(G_n, G_k), \quad \text{for all } 1 \le k \le n-1.$$

Therefore, iterating the argument above, we see that every vector v_T of the Gelfand-Tsetlin basis is an eigenvector for $\rho(f)$, as desired. □

Exercises.

2.2.2. Suppose $H \le G$. Verify that, for $f_1, f_2 \in L(H)$, we have $(f_1 * f_2)^G_H = (f_1)^G_H * (f_2)^G_H$.

2.2.3 ([CSST10, Ex. 2.2.5]). Prove that if the functions $f_1, \dots, f_n \in \mathrm{GZ}(n)$ separate the vectors of the GZ-bases, then the set $\{\delta_{1_G}, f_1, f_2, \dots, f_n\}$ generates $\mathrm{GZ}(n)$ as an algebra. *Hint*: For $\rho \in \widehat{G_n}$ and $T \in \mathcal{T}(\rho)$, define $F_{\rho,T}$ to be the convolution of all

$$\frac{f_i - \alpha_{\sigma,S,i}\delta_{1_G}}{\alpha_{\rho,T,i} - \alpha_{\sigma,S,i}}, \quad \sigma \in \widehat{G_n},\ S \in \mathcal{T}(\sigma),\ 1 \le i \le n,\ \text{such that} (\sigma, S) \ne (\rho, T) \text{ and } \alpha_{\sigma,S,i} \ne \alpha_{\rho,T,i},$$

using the notation of (2.14). (Note that the order of convolution is irrelevant since $\mathrm{GZ}(n)$ is commutative.) Show that $F_{\rho,T}$ is given by (2.13).

Chapter 3

The Okounkov–Vershik approach

In this chapter, we study the representation theory of the symmetric groups following the approach of Okounkov and Vershik [OV96, VO04, Ver06]. We closely follow the presentation in [CSST10, Ch. 3]. The reference [Py03] may also be helpful to the reader.

3.1 The Young poset

In this section we introduce some algebraic and combinatorial concepts that will be used in our study of the representation theory of the symmetric group.

3.1.1 Partitions and conjugacy classes in $\mathfrak{S}_n$

Definition 3.1.1. Suppose n is a positive integer. A *partition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ of positive integers such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_h$ such that $\lambda_1 + \lambda_2 + \cdots \lambda_h = n$. We call h the *length* of the partition λ, and denote it $\ell(\lambda)$. We adopt the convention that $\lambda_r = 0$ for $r > h$. We write $\lambda \vdash n$ to indicate that λ is a partition of n. We also call $|\lambda| := n$ the *size* of λ.

Recall that $\mathfrak{S}_n$ is the symmetric group of all permutations of the set $\{1, 2, \dots, n\}$. A permutation $\gamma \in \mathfrak{S}_n$ is called a *cycle* of length t if there exist pairwise distinct elements $a_1, a_2, \dots, a_t \in \{1, 2, \dots, n\}$ such that

$$\gamma(a_i) = a_{i+1},\ 1 \le i \le t-1, \quad \gamma(a_t) = a_1,$$
$$\gamma(b) = b, \quad b \in \{1, 2, \dots, n\} \setminus \{a_1, a_2, \dots, a_t\}.$$

We denote this cycle by $\gamma = (a_1, a_2, \dots, a_t)$, or sometimes by

$$(a_1 \to a_2 \to \cdots a_t \to a_1). \tag{3.1}$$

A *transposition* is a cycle of length two.

Two cycles $\gamma = (a_1, \dots, a_t)$ and $\theta = (b_1, \dots, b_s)$ are *disjoint* if

$$\{a_1, \dots, a_r\} \cap \{b_1, \dots, b_s\} = \varnothing.$$

If γ and θ are disjoint cycles, then they commute: $\gamma\theta = \theta\gamma$.

Every $\pi \in \mathfrak{S}_n$ can be written as a product of disjoint cycles

$$\pi = (a_{1,1}, a_{1,2}, \ldots, a_{1,\mu_1})(a_{2,1}, a_{2,2}, \ldots, a_{2,\mu_2}) \cdots (a_{k,1}, a_{k,2}, \ldots, a_{k,\mu_k}), \tag{3.2}$$

where the list

$$a_{1,1}, \ldots, a_{1,\mu_1}, \ldots, a_{k,1}, \ldots, a_{k,\mu_k}$$

is a permutation of $1, 2, \ldots, n$. In particular $\sum_{i=1}^{k} \mu_i = n$. Rearranging the cycles if necessary, we may assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$. The partition $\mu = (\mu_1, \ldots, \mu_k)$ is called the *cycle type* of π. The expression (3.2) is called a *cycle decomposition* of π. It is unique up to cyclic permutation of the elements of the cycles and permutation of cycles of equal size.

We say that $i \in \{1, 2, \ldots, n\}$ is a *fixed point* of $\pi \in \mathfrak{S}_n$ if $\pi(i) = i$. Thus i is a fixed point of π if and only if i appears in a cycle of length 1 in the cycle decomposition of π.

If $\sigma \in \mathfrak{S}_n$, then

$$\sigma\pi\sigma^{-1} = \big(\sigma(a_{1,1}), \ldots, \sigma(a_{1,\mu_1})\big)\big(\sigma(a_{2,1}), \ldots, \sigma(a_{2,\mu_2})\big)\big(\sigma(a_{k,1}), \ldots, \sigma(a_{k,\mu_k})\big).$$

It follows that two elements $\pi, \pi' \in \mathfrak{S}_n$ are conjugate if and only if they have the same cycle type.

Proposition 3.1.2. *The conjugacy classes of $\mathfrak{S}_n$ are parameterized by partitions of n. The conjugacy class associated to $\lambda \vdash n$ consists of all permutations of cycle type λ.*

Proof. This follows from the above discussion. □

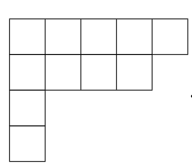
Exercises.

3.1.1. Let $\lambda \vdash n$. Deduce an explicit formula for the number of permutations of cycle type λ.

3.1.2 Young diagrams

Let $\lambda = (\lambda_1, \ldots, \lambda_h) \vdash n$. The *Young diagram* associated to λ, also called the *Young diagram of shape* λ, is the array consisting of n boxes, with h left-justified rows, the i-th row containing λ_i boxes for $1 \leq i \leq h$. It follows that the Young diagram has λ_1 columns. We will often abuse terminology and refer to λ itself as a Young diagram.

For example, the Young diagram associated to the partition $\lambda = (5, 4, 1, 1) \vdash 11$ is



.

The box in row i (with row 1 at the top of the diagram) and column j (with column 1 on the left of the diagram) will be said to be in position (i, j).

We say that a box in position (i, j) is *removable* if removing this box results in a Young diagram. In other words, it is removable if there are no boxes in position $(i+1, j)$ and $(i, j+1)$. Similarly, the position (i, j) is *addable* if we can add a box in position in position (i, j) and end up with a Young diagram. In other words, it is addable if

$$\lambda_i = j - 1 < \lambda_{i-1} \text{ or } (i = h + 1 \text{ and } j = 1).$$

Exercises.

3.1.2. For a Young diagram λ, let $a(\lambda)$ be the number of addable positions of λ and let $r(\lambda)$ be the number of removable boxes of λ. Show that $a(\lambda) - r(\lambda) = 1$.

3.1.3 Young tableaux

Suppose $\lambda \vdash n$. A *(bijective) Young tableau* of shaped λ is a bijection between the boxes of the Young diagram of shape λ and the set $\{1, 2, \ldots, n\}$. It is depicted by filling the boxes of the Young diagram with the numbers $1, 2, \ldots, n$, with exactly one number in each box. For example,

4	5	1	6	7
2	3	8		
9				

(3.3)

is a Young tableau of shape $(5, 3, 1)$. The plural of Young tableau is *Young tableaux*.

A Young tableau is *standard* if the numbers in the boxes are increasing along the rows (from left to right) and down the columns (from top to bottom). For instance, the Young tableau of (3.3) is not standard, while

1	3	5	8	9
2	4	7		
6				

is.

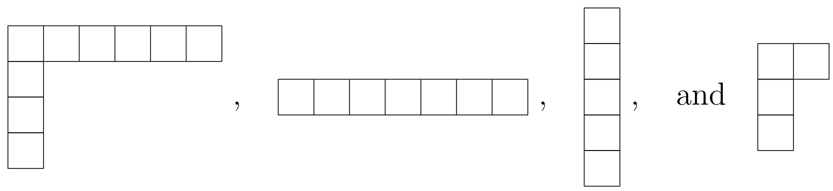
Note that in a standard Young tableau, the number 1 is always in position $(1, 1)$, while n is always in a removable box.

We denote by $\mathrm{Tab}(\lambda)$ the set of all standard tableaux of shape λ, and we let

$$\mathrm{Tab}(n) = \bigcup_{\lambda \vdash n} \mathrm{Tab}(\lambda).$$

Exercises.

3.1.3. A Young diagram is called a *hook* if every row below the first has one box. For example,



are hooks. Let λ be a hook. Deduce an explicit expression for the number of standard tableaux of shape λ.

3.1.4 Coxeter generators

The elements

$$s_i = (i, i+1) \in \mathfrak{S}_n, \quad i = 1, 2, \ldots, n-1,$$

are called *simple transpositions* (or *adjacent transpositions*). They are also called the *Coxeter generators* of $\mathfrak{S}_n$. (This is because of the connection to the more general theory of *Coxeter groups*.) The term "generator" will be justified in Proposition 3.1.3.

If T is a Young tableaux with n boxes and $\pi \in \mathfrak{S}_n$, then πT will denote the tableau obtained from T by replacing i with $\pi(i)$ for $i = 1, 2, \ldots, n$. For example, if $\pi = (156)(78)(234)(9)$ and

$$T = \begin{array}{|c|c|c|c|c|}\hline 4 & 5 & 1 & 6 & 7 \\ \hline 2 & 3 & 8 \\ \cline{1-3} 9 \\ \cline{1-1}\end{array}\ , \quad \text{then} \quad \pi T = \begin{array}{|c|c|c|c|c|}\hline 2 & 6 & 5 & 1 & 8 \\ \hline 3 & 4 & 7 \\ \cline{1-3} 9 \\ \cline{1-1}\end{array}\ .$$

If T is standard, then an *admissible transposition* for T is a simple transposition s_i such that $s_i T$ is standard. Thus, s_i is admissible for T if and only if i and $i+1$ belong neither to the same row, nor the same column of T.

An *inversion* for $\pi \in \mathfrak{S}_n$ is a pair (i, j) with $1 \leq i, j \leq n$ such that

$$i < j \quad \text{and} \quad \pi(i) > \pi(j).$$

Let $\mathcal{I}(\pi)$ denote set of all inversions for π and let

$$\ell(\pi) = |\mathcal{I}(\pi)|$$

denote the number of inversions of π.

The *length* of $\pi \in \mathfrak{S}_n$ is the smallest integer k such that π can be written as a product of k simple transpositions: $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$.

Proposition 3.1.3. *The length of $\pi \in \mathfrak{S}_n$ is equal to $\ell(\pi)$.*

Proof. We first claim that, for any $\pi \in \mathfrak{S}_n$, we have

$$\ell(\pi s_i) = \begin{cases} \ell(\pi) - 1 & \text{if } (i, i+1) \in \mathcal{I}(\pi), \\ \ell(\pi) + 1 & \text{if } (i, i+1) \notin \mathcal{I}(\pi). \end{cases} \tag{3.4}$$

First consider k satisfying $1 \le k < i$. There are three possibilities:

- $\pi(k) < \min\{\pi(i), \pi(i+1)\}$,
- $\pi(k) > \max\{\pi(i), \pi(i+1)\}$,
- $\min\{\pi(i), \pi(i+1)\} < \pi(k) < \max\{\pi(i), \pi(i+1)\}$.

In the first case, (k, i) and $(k, i+1)$ are neither inversions for π nor for πs_i. In the second case, (k, i) and $(k, i+1)$ are inversions for both. In the third case, exactly one of (k, i) and $(k, i+1)$ is an inversion for π, while only the other one is an inversion for πs_i. Thus, in all three cases, the number of inversions in the set $\{(k, i), (k, i+1)\}$ is the same for π and πs_i. A similar argument gives the same result for the case that $i+1 < k \le n$. Since $(i, i+1) \in \mathcal{I}(\pi)$ if and only if $(i, i+1) \notin \mathcal{I}(\pi s_i)$, the claim follows.

Now suppose $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$ is a minimal representation of π as a product of simple transpositions. Then, by (3.4), we have

$$\ell(\pi) = \ell(s_{i_1} s_{i_2} \cdots s_{i_{k-1}}) \pm 1 \le \ell(s_{i_1} s_{i_2} \cdots s_{i_{k-1}}) + 1 \le \cdots \le k.$$

Therefore, the length of π is greater than or equal to $\ell(\pi)$.

It remains to prove the reverse inequality, which we do by induction on n. It is clear for $n = 2$ that $\pi \in \mathfrak{S}_2$ can be written as a product of $\ell(\pi)$ transpositions. (Simply consider the two cases $\pi = (1)(2)$ and $\pi = (12)$.) Now suppose $n \ge 3$ and that any $\pi \in \mathfrak{S}_{n-1}$ can be written as a product of $\ell(\pi)$ transpositions.

Fix $\pi \in \mathfrak{S}_n$. Let $j_n = \pi^{-1}(n)$ and let

$$\pi_n = \pi s_{j_n} s_{j_n+1} \cdots s_{n-1}.$$

Thus $\pi_n(n) = n$ and so, by (3.4), we have

$$\ell(\pi_n s_{n-1}) = \ell(\pi_n) + 1.$$

Now, since $\pi_n s_{n-1}(n-1) = n$, we also have, by (3.4),

$$\ell(\pi_n s_{n-1} s_{n-2}) = \ell(\pi_n s_{n-1}) + 1 = \ell(\pi_n) + 2.$$

Continuing in this way, we see that

$$\ell(\pi_n) = \ell(\pi) - (n - j_n).$$

Now, since $\pi_n(n) = n$, it can naturally be viewed as an element of $\mathfrak{S}_{n-1}$. By our induction hypothesis, it can be written as a product of $\ell(\pi_n)$ simple transpositions. But then, since $\pi = \pi_n s_{n-1} s_{n-2} \cdots s_{j_n}$, we have that π can be written as a product of $\ell(\pi_n) + (n - j_n) = \ell(\pi)$ transpositions. This completes the proof of the induction step. $\square$

For $\lambda \vdash n$, let T^λ denote the standard tableau of shape λ, where we number the boxes $1, 2, \ldots, n$ from left-to-right starting in the top row, then continuing in the second row, etc. For example,

$$\text{if } \lambda = (5,3,1) \quad \text{then} \quad T^\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 \\ \cline{1-3} 9 \\ \cline{1-1} \end{array}.$$

For $T \in \mathrm{Tab}(\lambda)$, we let $\pi_T \in \mathfrak{S}_n$ denote the unique permutation such that $\pi_T T = T^\lambda$.

Theorem 3.1.4. *Suppose $T \in \mathrm{Tab}(\lambda)$. Then there exists a sequence of $\ell(\pi_T)$ admissible transpositions transforming T into T^λ.*

Proof. We prove the result by induction on $|\lambda|$. If $|\lambda| = 1$, then T^λ is the only tableau of shape λ and we are done. Now suppose $|\lambda| > 1$ and that the result holds for all tableau with fewer than $|\lambda|$ boxes.

Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ and $T \in \mathrm{Tab}(\lambda)$. Let j denote the entry in the rightmost box of the bottom row of T. If $j = n$ then, since the box is removable, we can consider the standard tableau T' of shape $\lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_k - 1)$ obtained by removing that box. By our induction hypothesis, there exists a sequence of $\ell(\pi_{T'})$ admissible transpositions transforming T' into $T^{\lambda'}$. This same sequence transforms T into T^λ and $\ell(\pi_{T'}) = \ell(\pi_T)$.

Now suppose $j \neq n$. Then s_j is admissible for T. Similarly, s_{j+1} is admissible for $s_j T, \ldots,$ s_{n-1} is admissible for $s_{n-2} \ldots s_{j_1} s_j T$. Now, $T'' = s_{n-1} s_{n-2} \cdots s_j T$ contains n in the rightmost box of the bottom row. Therefore, by the previous case, there exists a sequence of $\ell_{\pi_{T''}}$ simple transpositions transforming T'' into T^λ. It follows from (3.4) that $\ell(\pi_T) = \ell_{\pi_{T''}} + n - j$, completing the proof of the induction step. □

Corollary 3.1.5. *For any $T, S \in \mathrm{Tab}(\lambda)$, there is a sequence of admissible transpositions transforming S into T.*

Remark 3.1.6. Note that the proof of Theorem 3.1.4 gives a *standard* procedure to write π_T as a product of $\ell(\pi_T)$ admissible transpositions. We shall use this procedure in what follows.

Exercises.

3.1.4. Show that the simple transpositions in $\mathfrak{S}_n$ satisfy the relations

$$\begin{aligned} s_i^2 &= 1, && 1 \le i \le n-1, \\ s_i s_j &= s_j s_i, && 1 \le i, j \le n-1,\ |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, && 1 \le i \le n-2. \end{aligned}$$

(In fact, the above form a complete set of relations for the simple transpositions. That is, $\mathfrak{S}_n$ is the group with generators s_i, $1 \le i \le n-1$, and relations as given above.)

3.1.5 The content of a tableau

Suppose $T \in \mathrm{Tab}(\lambda)$ for some $\lambda \vdash n$. For $1 \le t \le n$, we let $\mathrm{row}(t)$ and $\mathrm{col}(t)$ denote the row and column of the box of T containing t. For example,

$$\text{if } T = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 7 \\ \hline 3 & 6 & 8 \\ \cline{1-3} 9 \\ \cline{1-1} \end{array} \quad \text{then} \quad \mathrm{row}(8) = 2,\ \mathrm{col}(8) = 3.$$

Given a box in the Young diagram λ with coordinates (i, j) we define the *content* of the box to be

$$c(i, j) := j - i.$$

For example, the contents of the boxes in $\lambda = (5, 3, 1)$ are as indicated:

$$\begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 4 \\ \hline -1 & 0 & 1 \\ \cline{1-3} -2 \\ \cline{1-1} \end{array}$$

So, essentially, the content of a box corresponds to the diagonal on which it lies.

We define the *content* of the tableau T to be

$$C(T) := \big(c(\mathrm{row}(1), \mathrm{col}(1)), c(\mathrm{row}(2), \mathrm{col}(2)), \dots, c(\mathrm{row}(n), \mathrm{col}(n))\big) \in \mathbb{Z}^n.$$

For example,

$$\text{if } T = \begin{array}{|c|c|c|c|c|} \hline 4 & 5 & 1 & 6 & 7 \\ \hline 2 & 3 & 8 \\ \cline{1-3} 9 \\ \cline{1-1} \end{array} \quad \text{then} \quad c(T) = (2, -1, 0, 0, 1, 3, 4, 1, -2).$$

Note that, for a fixed partition λ, the entries of the contents of the tableau of shape λ are the same, but potentially in different orders (i.e. they are permutations of one another).

Definition 3.1.7 ($\mathrm{Cont}(n)$)**.** Let $\mathrm{Cont}(n)$ be the set of all $(a_1, a_2, \dots, a_n) \in \mathbb{C}^n$ such that

(a) $a_1 = 0$,

(b) $\{a_j + 1, a_j - 1\} \cap \{a_1, a_2, \dots, a_{j-1}\} \neq \varnothing$ for all $j > 1$,

(c) if $a_i = a_j$ for some $i < j$ then $\{a_j - 1, a_j + 1\} \subseteq \{a_{i+1}, a_{i+2}, \dots, a_{j-1}\}$.

It follows from the definition that, in fact, $\mathrm{Cont}(n) \subseteq \mathbb{Z}^n$. For example,

$$\mathrm{Cont}(1) = \{0\} \quad \text{and} \quad \mathrm{Cont}(2) = \{(0, 1), (0, -1)\}. \tag{3.5}$$

For $\alpha, \beta \in \mathrm{Cont}(n)$, we will write $\alpha \approx \beta$ if $\pi\beta = \alpha$ for some $\pi \in \mathfrak{S}_n$ (here $\mathfrak{S}_n$ acts on $\mathbb{Z}^n$ by permuting the entries). Note that $\approx$ is an equivalence relation on $\mathrm{Cont}(n)$, but that $\mathfrak{S}_n$ does not act on $\mathrm{Cont}(n)$ since there are $\alpha \in \mathrm{Cont}(n)$ and $\pi \in \mathfrak{S}_n$ such that $\pi\alpha \notin \mathrm{Cont}(n)$. For example, if $\pi = (1, 2) \in \mathfrak{S}_2$, then

$$\alpha = (0, 1) \in \mathrm{Cont}(2) \quad \text{but} \quad \pi(0, 1) = (1, 0) \notin \mathrm{Cont}(2).$$

Theorem 3.1.8. *For any $T \in \mathrm{Tab}(n)$, we have $C(T) \in \mathrm{Cont}(n)$. Furthermore, the map*

$$\mathrm{Tab}(n) \to \mathrm{Cont}(n), \quad T \mapsto C(T),$$

is a bijection. In addition, for $T, S \in \mathrm{Tab}(n)$, we have

$$C(T) \approx C(S) \iff T \text{ and } S \text{ are tableaux of the same shape.}$$

Proof. A proof of this theorem can be found in [CSST10, Th. 3.1.10]. □

For $\alpha \in \mathrm{Cont}(n)$, we say that a simple transposition s_i is *admissible* for α if it is admissible for the unique $T \in \mathrm{Tab}(n)$ such that $\alpha = C(T)$.

Corollary 3.1.9. *For $\alpha, \beta \in C(T)$, we have $\alpha \approx \beta$ if and only if there exists a sequence of admissible transpositions transforming α into β.*

Proof. This follows from Corollary 3.1.5 and Theorem 3.1.8. □

Corollary 3.1.10. *The cardinality of the quotient set $\mathrm{Cont}(n)/\approx$ is equal to the number of partitions of n.*

Proof. This follows from Theorem 3.1.8. □

Exercises.

3.1.5. Suppose that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathrm{Cont}(n)$ and $1 \leq i \leq n-1$. Prove that s_i is admissible for α if and only if $a_{i+1} \neq a_i \pm 1$.

3.1.6 The Young poset

Let

$$\mathbb{Y} = \{\lambda : \lambda \vdash n, \ n \in \mathbb{Z}_{>0}\}$$

be the set of all partitions. Equivalently, $\mathbb{Y}$ is the set of all Young diagrams. We define a partial ordering on $\mathbb{Y}$ by saying that, for $\mu, \lambda \in \mathbb{Y}$, $\mu \preceq \lambda$ if and only if the Young diagram of μ is contained in the Young diagram of λ. In other words, if $\mu = (\mu_1, \ldots, \mu_k) \vdash n$ and $\lambda = (\lambda_1, \ldots, \lambda_h) \vdash m$, then

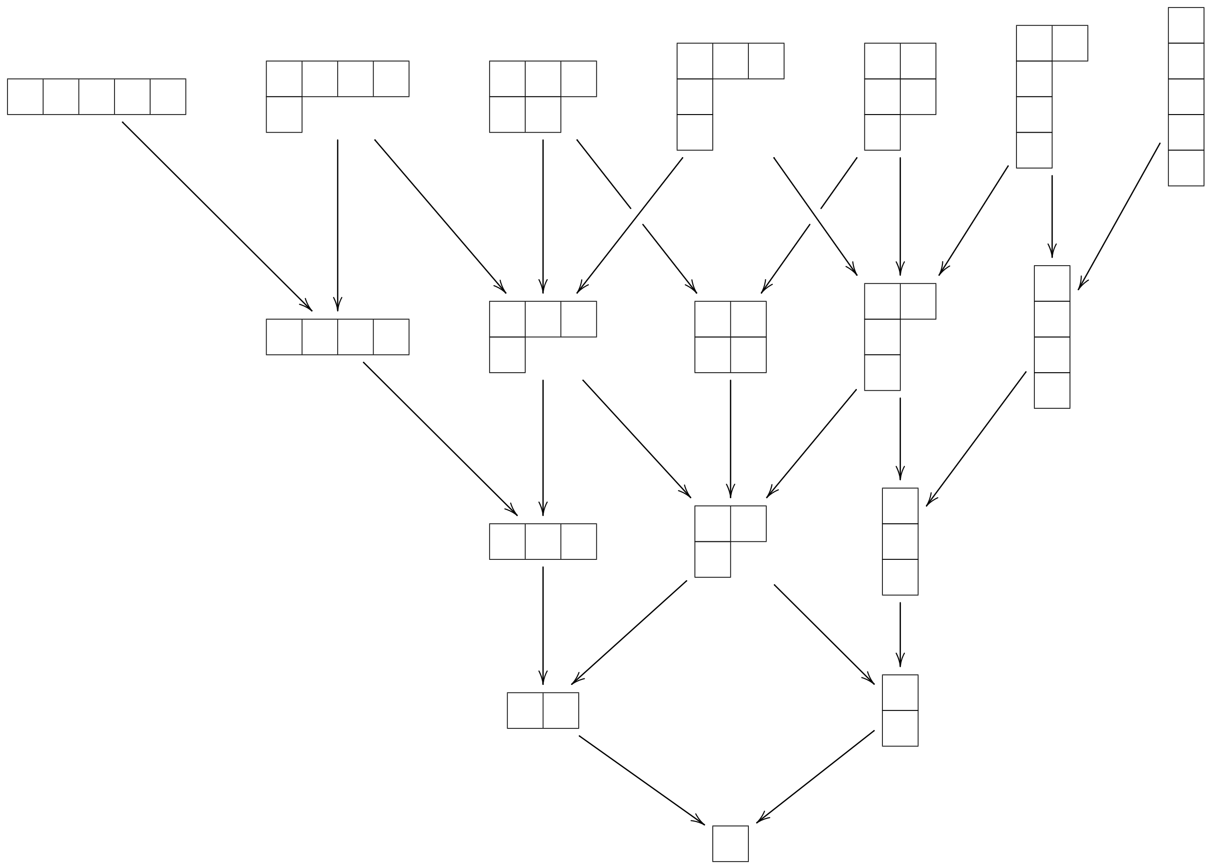
$$\mu \preceq \lambda \iff m \geq n \text{ and } \lambda_j \geq \mu_j \text{ for all } j = 1, 2, \ldots, k.$$

For example,

$$(4, 3, 1) \preceq (5, 3, 1, 1).$$

For $\mu, \lambda \in \mathbb{Y}$, we say that λ *covers* μ if μ is obtained from λ by removing a single box. In particular, this implies that $\mu \preceq \lambda$.

The *Hasse diagram* of $\mathbb{Y}$, also called the *Young (branching) graphs*, is the oriented graph with vertex set $\mathbb{Y}$ and an arrow from λ to μ if and only if λ covers μ. The bottom of the Young graph is as follows:



(One sometimes includes the empty Young diagram $\varnothing$ at the bottom of the graph.)

A *path* in the Young graph is a sequence

$$p = \left(\lambda^{(n)} \to \lambda^{(n-1)} \to \cdots \to \lambda^{(1)}\right)$$

of partitions $\lambda^{(k)} \vdash k$ such that $\lambda^{(k)}$ covers $\lambda^{(k-1)}$ for $k = 2, 3, \ldots, n$. We call $\ell(p) := n$ the *length* of the path p. We let $\Pi_n(\mathbb{Y})$ denote the set of all paths in the Young graph of length n and let

$$\Pi(\mathbb{Y}) = \bigcup_{n=1}^{\infty} \Pi_n(\mathbb{Y}).$$

To any standard Young tableau T of shape $\lambda \vdash n$ we can associate a path

$$\lambda = \lambda^{(n)} \to \lambda^{(n-1)} \to \cdots \to \lambda(1)$$

by letting $\lambda^{(k)}$, $1 \leq k \leq n$, be the Young diagram formed by the boxes of T with labels less than or equal to k. For example, to the standard tableau

1	2	5	6
3	4	7	
8			

we associate the path

$$(4,3,1) \to (4,3) \to (4,2) \to (3,2) \to (2,2) \to (2,1) \to (2) \to (1).$$

In this way we have a natural bijection

$$\Pi_n(\mathbb{Y}) \leftrightarrow \operatorname{Tab}(n) \tag{3.6}$$

which gives a bijection

$$\Pi(\mathbb{Y}) \leftrightarrow \bigcup_{n=1}^{\infty} \operatorname{Tab}(n). \tag{3.7}$$

Combining (3.6) with the bijection in Theorem 3.1.8 yields a bijection

$$\Pi_n(\mathbb{Y}) \leftrightarrow \operatorname{Cont}(n). \tag{3.8}$$

Proposition 3.1.11. *Suppose $\alpha, \beta \in \operatorname{Cont}(n)$ correspond to the paths*

$$\lambda^{(n)} \to \lambda^{(n-1)} \to \cdots \to \lambda^{(1)} \quad \textit{and} \quad \mu^{(n)} \to \mu^{(n-1)} \to \cdots \to \mu^{(1)},$$

respectively. Then $\alpha \approx \beta$ if and only if $\lambda^{(n)} = \mu^{(n)}$.

Proof. This follows immediately from Theorem 3.1.8. □

Exercises.

3.1.6. Suppose λ is a hook (see Exercise 3.1.3). How many paths in the Young graph are there that start at λ?

3.2 The Young–Jucys–Murphy elements and a Gelfand–Tsetlin basis for $\mathfrak{S}_n$

Our goal in this section is to prove that the chain

$$\mathfrak{S}_1 \le \mathfrak{S}_2 \le \cdots \le \mathfrak{S}_n \le \mathfrak{S}_{n+1} \le \cdots$$

is multiplicity free. This allows use to use the techniques of Chapter 2. In particular, we will study the Gelfand–Tsetlin algebra associated to this chain.

3.2.1 The Young–Jucys–Murphy elements

In the remainder of these notes, we will identity the Dirac function δ_π, $\pi \in \mathfrak{S}_n$, with π. Thus, elements $f \in L(\mathfrak{S}_n)$ will be written as formal sums

$$f = \sum_{\pi \in \mathfrak{S}_n} f(\pi)\pi.$$

Similarly, the characteristic function of $A \subseteq \mathfrak{S}_n$ will be denoted simply by A, so that

$$A = \sum_{\pi \in A} \pi.$$

In addition, the convolution of $f_1, f_2 \in L(\mathfrak{S}_n)$ will be denoted $f_1 \cdot f_2$ and written as a product of formal sums:

$$f_1 \cdot f_2 = \sum_{\pi \in \mathfrak{S}_n} \left(\sum_{\substack{\sigma,\theta \in \mathfrak{S}_n: \\ \sigma\theta = \pi}} f_1(\sigma) f_2(\theta) \right) \pi.$$

The *Young–Jucys–Murphy (YJM) elements* of $L(\mathfrak{S}_n)$ are defined by

$$X_1 = 0, \quad X_k = (1,k) + (2,k) + \cdots + (k-1,k), \; k = 2, \ldots, n.$$

In some places in the literature, these elements are simply called *Jucys–Murphy elements*.

Exercises.

3.2.1. Show that the following relations hold in $L(\mathfrak{S}_n)$:

$$\begin{aligned} X_{i+1} s_i &= s_i X_i + 1, & 1 \le i \le n-1, \\ X_j s_i &= s_i X_j, & 1 \le i \le n-1,\; 1 \le j \le n,\; j \ne i, i+1. \end{aligned}$$

3.2.2. Show that $X_i X_j = X_j X_i$ for all $1 \le i, j \le n$.

3.2.3. Fix $n \ge 2$. The *degenerate affine Hecke algebra* $\mathcal{H}_n$ is the $\mathbb{C}$-algebra generated by elements

$$t_i, \quad 1 \le i \le n-1, \quad \text{and} \quad x_j, \quad 1 \le j \le n,$$

subject to the relations

$$\begin{aligned} t_i^2 &= 1, & 1 \le i \le n-1, \\ t_i t_j &= t_j t_i, & 1 \le i, j \le n-1,\; |i-j| > 1, \\ t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, & 1 \le i \le n-2. \\ x_{i+1} t_i &= t_i x_i + 1, & 1 \le i \le n-1, \\ x_j t_i &= t_i x_j, & 1 \le i \le n-1,\; 1 \le j \le n,\; j \ne i, i+1, \end{aligned}$$

$$x_i x_j = x_j x_i, \qquad 1 \le i, j \le n.$$

(This means that for any $\mathbb{C}$-algebra $\mathcal{A}$ with elements t_i', x_j' satisfying the above relations, there is a unique algebra homomorphism $\mathcal{H}_n \to \mathcal{A}$ such that $t_i \mapsto t_i'$ and $x_j \mapsto x_j'$.) There is an injective algebra homomorphism $L(\mathfrak{S}_n) \hookrightarrow \mathcal{H}_n$ given by $s_i \mapsto t_i$ and we use this to view $L(\mathfrak{S}_n)$ as a subalgebra of $\mathcal{H}_n$.

Let I be the ideal of $\mathcal{H}_n$ generated by x_1. (See Exercise 1.2.8.) Prove that $\mathcal{H}_n/I \cong L(\mathfrak{S}_n)$, as algebras (or as rings). You may use the fact that $\mathcal{H}_n$ has a basis given by the elements

$$x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} \pi, \quad k_1, k_2, \dots, k_n \in \mathbb{Z}_{\ge 0},\ \pi \in \mathfrak{S}_n, \tag{3.9}$$

where we view the element $\pi \in \mathfrak{S}_n$ (identified with $\delta_\pi \in L(\mathfrak{S}_n)$) as an element of $\mathcal{H}_n$ as explained above.

3.2.2 Marked permutations

Fix $\ell, k \le 1$. In this section, we let $\mathfrak{S}_{\ell+k}$, $\mathfrak{S}_\ell$, and $\mathfrak{S}_k$ denote the symmetric groups on the sets $\{1, 2, \dots, \ell + k\}$, $\{1, 2, \dots, \ell\}$ and $\{\ell + 1, \ell + 2, \dots, \ell + k\}$, respectively. Thus

$$\mathfrak{S}_\ell, \mathfrak{S}_k \le \mathfrak{S}_{\ell+k} \quad \text{and} \quad \mathfrak{S}_\ell \cap \mathfrak{S}_k = \{1\}.$$

We let

$$Z(\ell, k) = \mathcal{C}(\mathfrak{S}_{\ell+k}, \mathfrak{S}_\ell)$$

denote the algebra of all $\mathfrak{S}_\ell$-conjugacy invariant functions in $L(\mathfrak{S}_{\ell+k})$.

We wish to analyze the algebra $Z(\ell, k)$. The first step is to parameterize the orbits of the $\mathfrak{S}_\ell$-conjugacy action on $\mathfrak{S}_{\ell+k}$. Recall that, for $\pi \in \mathfrak{S}_\ell$ and $\theta = \mathfrak{S}_{\ell+k}$, the cycle decomposition of $\pi\theta\pi^{-1}$ is obtained from the cycle decomposition of θ by replacing $1, 2, \dots, \ell$ with $\pi(1), \pi(2), \dots, \pi(\ell)$. Thus, the $\mathfrak{S}_\ell$-orbit of θ is obtained from its cycle decomposition by permuting in all possible ways the elements $1, 2, \dots, \ell$, leaving the remaining elements $\ell + 1, \ell + 2, \dots, \ell + k$ unchanged.

Consider the cycle decomposition of a permutation using our notation (3.1) for cycles:

$$(a_{1,1} \to a_{1,2} \to \cdots \to a_{1,\mu_1} \to a_{1,1})(a_{2,1} \to \cdots \to a_{2,\mu_2} \to a_{2,1}) \cdots (a_{r,1} \to \cdots \to a_{r,\mu_r} \to a_{r,1}). \tag{3.10}$$

A *marked permutation* is a permutation as in (3.10), together with a labeling of all the arrows by nonnegative integers, called *tags*, potentially will some additional empty cycles also labeled with tags. For example, the permutation (3.10) may be marked as follows:

$$\begin{aligned} (a_{1,1} \xrightarrow{u_{1,1}} a_2 \xrightarrow{u_{1,2}} \cdots \xrightarrow{u_{1,\mu_1-1}} a_{1,\mu_1} \xrightarrow{u_{1,\mu_1}} a_{1,1})(a_{2,1} \xrightarrow{u_{2,1}} \cdots \xrightarrow{u_{2,\mu_2-1}} a_{2,\mu_2} \xrightarrow{u_{2,\mu_2}} a_{2,1}) \\ \cdots (a_{r,1} \xrightarrow{u_{r,1}} \cdots \xrightarrow{u_{r,\mu_r-1}} a_{r,\mu_r} \xrightarrow{u_{r,\mu_r}} a_{r,1})(\xrightarrow{v_1})(\xrightarrow{v_2}) \cdots (\xrightarrow{v_s}). \end{aligned} \tag{3.11}$$

The orbits of the conjugacy action of $\mathfrak{S}_\ell$ on $\mathfrak{S}_{\ell+k}$ are in natural one-to-one correspondence with the set of all marked permutations of $\{\ell+1, \dots, \ell+k\}$ such that the sum of the tags is equal to ℓ. The orbit corresponding to a given marked permutation is obtained by inserting, in all possible ways, the elements $\{1, \dots, \ell\}$ into the marked permutation, with the number of elements added at any particular arrow equal to the label of that arrow.

Example 3.2.1. If $\ell = 12$ and $k = 8$, then the marked permutation

$$(19 \xrightarrow{2} 15 \xrightarrow{1} 13 \xrightarrow{0} 14 \xrightarrow{0} 19)(16 \xrightarrow{2} 20 \xrightarrow{1} 17 \xrightarrow{1} 16)(18 \xrightarrow{2} 18)(\xrightarrow{2})(\xrightarrow{1})$$

corresponds to the orbit consisting of all permutations of the form

$$(19 \to y_1 \to y_2 \to 15 \to y_3 \to 13 \to 14 \to 19)(16 \to y_4 \to y_5 \to 20 \to y_6 \to 17 \to y_7 \to 16) \\ \cdot (18 \to y_8 \to y_9 \to 18)(y_{10} \to y_{11} \to y_{10})(y_{12} \to y_{12}),$$

where $\{y_1, y_2, \ldots, y_{12}\} = \{1, 2, \ldots, 12\}$.

We will typically omit trivial cycles of the form $(a \xrightarrow{0} a)$ and $(\xrightarrow{1})$. Note that, when omitting such trivial cycles, the sum of the tags is $\leq \ell$, with strict inequality a possibility.

Theorem 3.2.2. *Let $\mathfrak{S}_{n-1}$ be the symmetric group on $\{1, 2, \ldots, n-1\}$. Then $(\mathfrak{S}_n \times \mathfrak{S}_{n-1}, \widetilde{\mathfrak{S}}_{n-1})$ is a symmetric Gelfand pair.*

Proof. In light of Proposition 2.1.9, it suffices to prove that every $\pi \in \mathfrak{S}_n$ is $\mathfrak{S}_{n-1}$-conjugate to π^{-1}. Suppose π has cycle decomposition

$$\pi = (n = a_{1,1} \to a_{1,2} \to \cdots \to a_{1,\mu_1} \to n)(a_{2,1} \to \cdots \to a_{2,\mu_2} \to a_{2,1}) \\ \cdots (a_{r,1} \to \cdots \to a_{r,\mu_r} \to a_{r,1}).$$

Then π belongs to the $\mathfrak{S}_{n-1}$-conjugacy class corresponding to the marked permutation

$$(n \xrightarrow{\mu_1 - 1} n)(\xrightarrow{\mu_2}) \cdots (\xrightarrow{\mu_r}).$$

Then

$$\pi^{-1} = (n \to a_{1,\mu_1} \to \cdots \to a_{1,1} = n)(a_{2,1} \to a_{2,\mu_2} \to \cdots \to a_{2,1}) \cdots (a_{r,1} \to a_{r,\mu_r} \cdots \to a_{r,1})$$

clearly belongs to the same conjugacy class. □

Corollary 3.2.3. *The algebra $\mathcal{C}(\mathfrak{S}_n, \mathfrak{S}_{n-1})$ is commutative, and $\mathfrak{S}_{n-1}$ is a multiplicity-free subgroup of $\mathfrak{S}_n$. Thus*

$$\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \cdots \leq \mathfrak{S}_{n-1} \leq \mathfrak{S}_n \leq \cdots$$

is a multiplicity-free chain.

Proof. This follows from Theorems 2.1.7 and 3.2.2. □

Example 3.2.4. (a) For $j = 1, \ldots, k$, the YJM element $X_{\ell+j}$ can be written as

$$X_{\ell+j} = (\ell + j \xrightarrow{1} \ell + j) + \sum_{h=\ell+1}^{\ell+j-1} (\ell + j \xrightarrow{0} h \xrightarrow{0} \ell + j). \tag{3.12}$$

(Recall our convention that, for $A \subseteq \mathfrak{S}_{\ell+k}$, we denote the characteristic function of A simply by A.) In particular, $X_{\ell+1}, X_{\ell+2}, \ldots, X_{\ell+k} \in Z(\ell, k)$ since these elements are all sums of characteristic functions of orbits under the conjugacy action of $\mathfrak{S}_\ell$.

(b) Any $\sigma \in \mathfrak{S}_k$ forms a one-element orbit of $\mathfrak{S}_\ell$. Thus, viewing σ as an element of $L(\mathfrak{S}_k)$, we have $\mathfrak{S}_k \subseteq Z(\ell, k)$.

(c) It is clear that $Z(\ell) := Z(L(\mathfrak{S}_\ell)) \subseteq Z(\ell, k)$.

It follows from Example 3.2.4 that

$$\langle X_{\ell+1}, X_{\ell+2}, \dots, X_{\ell+k}, \mathfrak{S}_k, Z(\ell)\rangle \subseteq Z(\ell, k), \tag{3.13}$$

where, on the left side, the angled brackets denote the subgroup of $L(\mathfrak{S}_{\ell+k})$ generated by the elements/sets inside the brackets.

Exercises.

3.2.4. For $\sigma \in \mathfrak{S}_k$, describe the marked permutation corresponding to the $\mathfrak{S}_\ell$-orbit $\{\sigma\}$ (see Example 3.2.4(b)).

3.2.5. Suppose that $\mathcal{C}_\lambda$ is the conjugacy class of $\mathfrak{S}_\ell$ corresponding to $\lambda \vdash \ell$ (see Proposition 3.1.2). Describe the marked permutation corresponding to the $\mathfrak{S}_\ell$-orbit formed by $\mathcal{C}_\lambda$ (see Example 3.2.4(c)).

3.2.3 Olshanskii's Theorem

The goal of this subsection is to prove the reverse of the inclusion (3.13). Precisely, we want to prove that $Z(\ell, k)$ is generated by $X_{\ell+1}, X_{\ell+2}, \dots, X_{\ell+k}$, $\mathfrak{S}_k$, and $Z(\ell)$.

Let $Z_h(\ell, k)$ be the subspace of $Z(\ell, k)$ spanned by the $\mathfrak{S}_\ell$-conjugacy classes consisting of permutations with at least $\ell + k - h$ fixed points (equivalently, moving at most h elements). Then we have a filtration

$$\mathbb{C}1 = Z_0(\ell, k) \subseteq Z_1(\ell, k) \subseteq Z_{\ell+k-1}(\ell, k) \subseteq Z_{\ell+k}(\ell, k) = Z(\ell, k). \tag{3.14}$$

We will essentially be interested in "leading terms" with respect to this filtration. More precisely, for $f_1, f_2, f_3 \in Z(\ell, k)$, we will write

$$f_1 \cdot f_2 = f_3 + \text{lower terms}$$

if there exists h such that

$$f_3 \in Z_h(\ell, k) - Z_{h-1}(\ell, k) \quad \text{and} \quad f_1 \cdot f_2 - f_3 \in Z_{h-1}(\ell, k).$$

Lemma 3.2.5. *Let $i, j \geq 1$. Suppose $a_1, a_2, \dots, a_i$ and $b_1, b_2, \dots, b_j$ are each sequences of distinct elements in $\{\ell+1, \dots, \ell+k\}$. (Note that we do not require the a_k to be distinct from the b_k.) Suppose that, in $\mathfrak{S}_k$, we have*

$$(a_1 \to a_2 \to \cdots \to a_i \to a_1)(b_1 \to b_2 \to \cdots \to b_j \to b_1) = (c_1 \to c_2 \to \cdots \to c_h \to c_1),$$

with

$$h = |\{a_1, a_2, \ldots, a_i\} \cup \{b_1, b_2, \ldots, b_j\}| \leq i + j.$$

Let $u_1, u_2, \ldots, u_i, v_1, v_2, \ldots, v_j \geq 0$ *such that*

$$u_1 + u_2 + \cdots + u_i + v_1 + v_2 + \cdots + v_j \leq \ell.$$

Then, in $Z(\ell, k)$*, we have*

$$\left(a_1 \xrightarrow{u_1} a_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{i-1}} a_i \xrightarrow{u_i} a_1\right)\left(b_1 \xrightarrow{v_1} b_2 \xrightarrow{v_2} \cdots \xrightarrow{v_{j-1}} b_j \xrightarrow{v_j} v_1\right)$$
$$= \left(c_1 \xrightarrow{w_1} c_2 \xrightarrow{w_2} \cdots \xrightarrow{w_{h-1}} c_h \xrightarrow{w_h} c_1\right) + \textit{lower terms},$$

where

$$w_s = \begin{cases} v_t & \textit{if } c_s = b_t \textit{ and } b_{t+1} \notin \{a_1, a_2, \ldots, a_i\}, \\ v_t + u_m & \textit{if } c_s = b_t \textit{ and } b_{t+1} = a_m, \\ u_m & \textit{if } c_s = a_m \notin \{b_1, b_2, \ldots, b_j\}. \end{cases} \tag{3.15}$$

for $1 \leq s \leq h$.

Proof. Consider a product of the form

$$(a_1, x_{1,1}, x_{1,2}, \ldots, x_{1,u_1}, a_2, x_{2,1}, \ldots, x_{2,u_2}, \ldots, a_i, x_{i,1}, \ldots, x_{i,u_i})$$
$$\cdot (b_1, y_{1,1}, \ldots, y_{1,v_1}, \ldots, b_j, y_{j,1}, \ldots, y_{j,v_j}). \tag{3.16}$$

If the numbers $x_{1,1}, \ldots, x_{i,u_i}, y_{1,1}, \ldots, y_{j,v_j}$ are all distinct, which is possible since $u_1 + \cdots + u_i + v_1 + \cdots + v_j \leq \ell$, then (3.16) is equal to a permutation of the form

$$(c_1, z_{1,1}, z_{1,2}, \ldots, z_{1,w_1}, c_2, \ldots, c_h, z_{h,1}, \ldots, z_{h,w_h}),$$

where $w_1, w_2, \ldots, w_h$ are given by (3.15).

Otherwise, the product (3.16) moves fewer than $h + w_1 + w_2 + \cdots + w_h$ elements. □

Example 3.2.6. (a) For $a \in \{\ell+1, \ell+2, \ldots, \ell+k\}$ and $0 \leq u \leq \ell$, we have

$$\left(a \xrightarrow{1} a\right)^u = \left(a \xrightarrow{u} a\right) + \text{lower terms.}$$

(b) For $a, b \in \{\ell+1, \ell+2, \ldots, \ell+k\}$, $a \neq b$, and $0 \leq u \leq \ell$, we have

$$\left(b \xrightarrow{u} b\right)\left(a \xrightarrow{0} b \xrightarrow{0} a\right) = \left(a \xrightarrow{u} b \xrightarrow{0} a\right).$$

Note that there are no lower order terms in this case.

(c) For $a_1, a_2, \ldots, a_i \in \{\ell+1, \ell+2, \ldots, \ell+k\}$, pairwise distinct, and $u_1, u_2, \ldots, u_i \geq 0$ with $u_1 + u_2 + \cdots u_i \leq \ell$, we have

$$\left(a_1 \xrightarrow{u_i} a_1\right)\left(a_1 \xrightarrow{u_{i-1}} a_i \xrightarrow{0} a_1\right) \cdots \left(a_1 \xrightarrow{u_2} a_3 \xrightarrow{0} a_1\right)\left(a_1 \xrightarrow{u_1} a_2 \xrightarrow{0} a_1\right)$$
$$= \left(a_1 \xrightarrow{u_1} a_2 \xrightarrow{u_2} a_3 \xrightarrow{u_3} \cdots \xrightarrow{u_{i-1}} a_i \xrightarrow{u_i} a_1\right) + \text{lower terms.}$$

Theorem 3.2.7 (Olshanskii's Theorem)**.** *The centralizer algebra $Z(\ell,k)$ is generated by the Young–Jucys–Murphy elements $X_{\ell+1}, X_{\ell+2}, \dots, X_{\ell+k}$, the subgroup $\mathfrak{S}_k$, and the center $Z(\ell)$ of $\mathfrak{S}_\ell$. In other words, we have*

$$Z(\ell,k) = \langle X_{\ell+1}, X_{\ell+2}, \dots, X_{\ell+k}, \mathfrak{S}_k, Z(\ell)\rangle.$$

Proof. Let

$$\mathcal{A} = \langle X_{\ell+1}, X_{\ell+2}, \dots, X_{\ell+k}, \mathfrak{S}_k, Z(\ell)\rangle.$$

We have $\mathcal{A} \subseteq Z(\ell,k)$ from (3.13).

Keeping in mind the filtration (3.14), we will prove by induction on h that $Z_h(\ell,k) \subseteq \mathcal{A}$ for all $h = 0,1,\dots,\ell+k$. Since $Z_0(\ell,k) = \mathbb{C}1 \subseteq \mathcal{A}$, our base case is proved.

Now suppose that $1 \le h \le \ell+k$, and that the result holds for $h-1$. For $a, j \in \{\ell+1, \ell+2, \dots, \ell+k\}$ with $a \neq j$, we have

$$\left(a \xrightarrow{0} j \xrightarrow{0} a\right) \in \mathfrak{S}_k.$$

Thus, by (3.12), we have

$$\left(a \xrightarrow{1} a\right) = X_a - \sum_{j=\ell+1}^{a-1} \left(a \xrightarrow{0} j \xrightarrow{0} a\right) \in \mathcal{A}.$$

Now, the typical orbit in $Z_h(\ell,k) - Z_{h-1}(\ell,k)$ is of the form

$$\begin{aligned}(a_{1,1} \xrightarrow{u_{1,1}} a_{1,2} \xrightarrow{u_{1,2}} \cdots \xrightarrow{u_{1,\mu_1-1}} a_{1,\mu_1} \xrightarrow{u_{1,\mu_1}} a_{1,1})(a_{2,1} \xrightarrow{u_{2,1}} \cdots \xrightarrow{u_{2,\mu_2-1}} a_{2,\mu_2} \xrightarrow{u_{2,\mu_2}} a_{2,1}) \\ \cdots (a_{r,1} \xrightarrow{u_{r,1}} \cdots \xrightarrow{u_{r,\mu_r-1}} a_{r,\mu_r} \xrightarrow{u_{r,\mu_r}} a_{r,1})(\xrightarrow{v_1})(\xrightarrow{v_2})\cdots(\xrightarrow{v_s}) \quad (3.17)\end{aligned}$$

for pairwise distinct elements

$$a_{1,1}, a_{1,2}, \dots, a_{1,\mu_1}, \dots, a_{r,1}, \dots, a_{r,\mu_r} \in \{\ell+1, \ell+2, \dots, \ell+k\}$$

and

$$u_{1,1}, u_{1,2}, \dots, u_{1,\mu_1}, \dots, u_{r,1}, \dots, u_{r,\mu_r}, v_1, v_2, \dots, v_s \in \mathbb{Z}_{\ge 0}$$
$$\text{such that} \quad \sum_{p=1}^{r}\sum_{q=1}^{\mu_p} u_{p,q} + \sum_{p=1}^{s} v_p = h \le \ell.$$

Repeated application of Lemma 3.2.5 (see also Example 3.2.6) gives that

$$\begin{aligned}&\left(a_{1,1} \xrightarrow{1} a_{1,1}\right)^{u_{1,\mu_1}} \left(a_{1,\mu_1} \xrightarrow{1} a_{1,\mu_1}\right)^{u_{1,\mu_1-1}} \left(a_{1,1} \xrightarrow{0} a_{1,\mu_1} \xrightarrow{0} a_{1,1}\right) \cdots \\ &\quad \cdots \left(a_{1,3} \xrightarrow{1} a_{1,3}\right)^{u_{1,2}} \left(a_{1,1} \xrightarrow{0} a_{1,3} \xrightarrow{0} a_{1,1}\right) \left(a_{1,2} \xrightarrow{1} a_{1,2}\right)^{u_{1,1}} \left(a_{1,1} \xrightarrow{0} a_{1,2} \xrightarrow{0} a_{1,1}\right) \cdots \\ &\quad \cdots \\ &\quad \cdots \left(a_{r,1} \xrightarrow{1} a_{r,1}\right)^{u_{r,\mu_r}} \left(a_{r,\mu_r} \xrightarrow{1} a_{r,\mu_r}\right)^{u_{r,\mu_r-1}} \left(a_{r,1} \xrightarrow{0} a_{r,\mu_r} \xrightarrow{0} a_{r,1}\right) \cdots\end{aligned}$$

$$\cdots \left(a_{r,3} \xrightarrow{1} a_{r,3}\right)^{u_{r,2}} \left(a_{r,1} \xrightarrow{0} a_{r,3} \xrightarrow{0} a_{r,1}\right) \left(a_{r,2} \xrightarrow{1} a_{r,2}\right)^{u_{r,1}} \left(a_{r,1} \xrightarrow{0} a_{r,2} \xrightarrow{0} a_{r,1}\right) \cdots$$
$$\cdots \left(\xrightarrow{v_1}\right) \left(\xrightarrow{v_2}\right) \cdots \left(\xrightarrow{v_s}\right)$$

is equal to (3.17) modulo terms. Thus, by the inductive hypothesis, we have that (3.17) is an element of $\mathcal{A}$, completing the proof. □

Corollary 3.2.8. *The Gelfand–Tsetlin algebra* $\mathrm{GZ}(n)$ *of the multiplicity free chain*

$$\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \cdots \leq \mathfrak{S}_n$$

is generated by the Young–Jucys–Murphy elements $X_1, X_2, \ldots, X_n$.

Proof. For $2 \leq k \leq n$, let T_k denote the set of all transpositions (not necessarily simple) in $\mathfrak{S}_k$. Then $T_k \in Z(k)$ (recall that we identify a subset of $\mathfrak{S}_k$ with its characteristic function). Thus

$$X_k = T_k - T_{k-1} \in \mathrm{GZ}(n).$$

Hence $\langle X_1, X_2, \ldots, X_n \rangle \in \mathrm{GZ}(n)$.

We now prove by induction on n that $\mathrm{GZ}(n) = \langle X_1, X_2, \ldots, X_n \rangle$. The result for $n = 1$ is trivial. Assume $n > 1$ and that the result holds for $n - 1$. Then

$$\begin{aligned} Z(n) &= \mathcal{C}(\mathfrak{S}_n, \mathfrak{S}_n) \\ &\subseteq \mathcal{C}(\mathfrak{S}_n, \mathfrak{S}_{n-1}) \\ &= Z(n-1, 1) \\ &= \langle Z(n-1), X_n \rangle, && \text{(by Theorem 3.2.7).} \end{aligned}$$

Thus, using the induction hypothesis, we have

$$\mathrm{GZ}(n) = \langle \mathrm{GZ}(n-1), Z(n) \rangle = \langle \mathrm{GZ}(n-1), X_n \rangle = \langle X_1, X_2, \ldots, X_{n-1}, X_n \rangle,$$

completing the proof of the induction step. □

Exercises.

3.2.6. Use Corollary 3.2.8 to give an alternate proof that $Z(n-1, 1) = \mathcal{C}(\mathfrak{S}_n, \mathfrak{S}_{n-1})$ is commutative (see Corollary 3.2.3).

3.3 The spectrum of the Young–Jucys–Murphy elements and the branching graph of $\mathfrak{S}_n$

In this section, our goal is to show that the spectrum (i.e. set of possible eigenvalues) of the YJM elements is given by $\mathrm{Cont}(n)$ (see Definition 3.1.7) and prove that the branching graph of the multiplicity-free chain

$$\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \cdots \leq \mathfrak{S}_n \leq \cdots$$

is precisely the Young graph.

3.3.1 The weight of a Young basis vector

For $\rho \in \widehat{\mathfrak{S}_n}$, we have the Gelfand–Tsetlin basis $\{v_T : T \in \mathcal{T}(\rho)\}$ associated with the multiplicity free chain $\mathfrak{S}_1 \leq \mathfrak{S}_2 \leq \cdots \leq \mathfrak{S}_n$. In this setting, we also call this basis the *Young basis* for V_ρ.

By Theorem 2.2.2 and Corollary 3.2.8, every v_T is an eigenvector for $\rho(X_j)$ for all $1 \leq j \leq n$. We define the *weight* of v_T to be

$$\alpha(T) = (a_1, a_2, \ldots, a_n), \quad \text{where } \rho(X_j) = a_j v_T,\ 1 \leq j \leq n. \tag{3.18}$$

Since $X_1, \ldots, X_n$ generate $\mathrm{GZ}(n)$ by Corollary 3.2.8, it follows from Corollary 2.2.3 that v_T is determined, up to a scalar factor, by $\alpha(T)$.

It follows from Theorem 1.5.7 that $\rho(X_j)$ is self-adjoint for all X_j and all representations ρ of $\mathfrak{S}_n$. (See Exercise 3.3.1.)

Proposition 3.3.1. *Suppose $\rho \in \widehat{\mathfrak{S}_n}$ and*

$$T = (\rho = \rho_n \to \rho_{n-1} \to \cdots \to \rho_2 \to \rho_1) \in \mathcal{T}(\rho).$$

Then $\rho(s_k)v_T$ is a linear combination of vectors $v_{T'}$ with T' of the form

$$T' = (\sigma = \sigma_n \to \sigma_{n-1} \to \cdots \to \sigma_2 \to \sigma_1) \in \mathcal{T}(\rho), \quad \textit{such that } \sigma_i = \rho_i \textit{ for all } i \neq k.$$

Proof. For $1 \leq j \leq n$, let V_j denote the representation space of ρ_j. Then

$$V_j = \{\rho_j(f)v_T : f \in L(\mathfrak{S}_j)\},$$

since the right side is a $\mathfrak{S}_j$-invariant subspace generated by a nonzero element of V_j.

If $j > k$, then $s_k \in \mathfrak{S}_j$, and so $\rho_j(s_k) \in V_j$. Thus

$$\sigma_j = \rho_j \quad \text{for all } j = k+1, k+2, \ldots, n.$$

Now suppose $j < k$. Then s_k and $\mathfrak{S}_j$ commute. If we define

$$W_j = \{\rho_j(f)\rho(s_k)v_T : f \in L(\mathfrak{S}_j)\} = \rho(s_k)V_j,$$

then we have an isomorphism of $\mathfrak{S}_j$-representations

$$V_j \to W_j \quad \rho_j(f)v_T \mapsto \rho_j(f)\rho(s_k)v_T.$$

Therefore $\rho(s_k)v_T$ belongs to the ρ_j-isotypic component of $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_j}\rho$, and so $\sigma_j = \rho_j$. $\square$

Exercises.

3.3.1. Use Theorem 1.5.7 to prove that $\rho(X_j)$ is self-adjoint for all $1 \leq j \leq n$ and all representations ρ of $\mathfrak{S}_n$.

3.3.2 The spectrum of the YJM elements

We define the *spectrum* of the YJM elements to be

$$\operatorname{Spec}(n) = \{\alpha(T) : T \in T(\rho),\ \rho \in \widehat{\mathfrak{S}_n}\},$$

where $\alpha(T)$ is the weight of v_T, as in (3.18). Since the elements of the Young basis are uniquely determined by their weight, we have

$$|\operatorname{Spec}(n)| = \sum_{\rho \in \widehat{\mathfrak{S}_n}} \dim V_\rho.$$

In particular, $\operatorname{Spec}(n)$ is in natural bijection with the set of all paths in the branching graph of the chain $\mathfrak{S}_1 \le \mathfrak{S}_2 \le \cdots \le \mathfrak{S}_n$. We let T_α denote the path corresponding to $\alpha \in \operatorname{Spec}(n)$, and we let v_α denote the Young basis vector corresponding to T_α.

We define an equivalence relation on $\operatorname{Spec}(n)$ by declaring that $\alpha \sim \beta$ if v_α and v_β belong to the same irreducible $\mathfrak{S}_n$-representation. Equivalently, $\alpha \sim \beta$ if the corresponding paths in the branching graph start at the same vertex. It follows that

$$|\operatorname{Spec}(n)/\sim| = \left|\widehat{\mathfrak{S}_n}\right|. \tag{3.19}$$

We would now like to deduce an explicit description of $\operatorname{Spec}(n)$ and $\sim$ using the relations (see Exercise 3.2.1)

$$X_{i+1}s_i = s_iX_i + 1, \qquad 1 \le i \le n-1, \tag{3.20}$$

$$X_js_i = s_iX_j, \qquad 1 \le i \le n-1,\ 1 \le j \le n,\ j \ne i, i+1. \tag{3.21}$$

Note that (3.20) is equivalent to

$$s_iX_{i+1} - 1 = X_is_i, \quad 1 \le i \le n-1,\ 1 \le j \le n,\ j \ne i, i+1. \tag{3.22}$$

In what follows, if v_α is a vector of the Young basis of an irreducible representation ρ of $\mathfrak{S}_n$, we denote $\rho(s_i)v_\alpha$ and $\rho(X_i)v_\alpha$ by s_iv_α and X_iv_α, respectively. (In other words, we will use the notation of *modules*, which is equivalent to that of representations.)

Proposition 3.3.2. *Suppose* $\alpha = (a_1, a_2, \dots, a_n) \in \operatorname{Spec}(n)$ *and* $1 \le i \le n-1$.

(a) $a_i \ne a_{i+1}$.

(b) $a_{i+1} = a_i \pm 1$ *if and only if* $s_iv_\alpha = \pm v_\alpha$.

(c) If $a_{i+1} \ne a_i \pm 1$, *then*

$$\alpha' := s_i\alpha = (a_1, a_2, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_n) \in \operatorname{Spec}(n),$$

$\alpha \sim \alpha'$, *and we have*

$$v_{\alpha'} = s_iv_\alpha - \frac{1}{a_{i+1} - a_i}v_\alpha \tag{3.23}$$

(up to a scalar factor). Moreover, the space $\langle v_\alpha, v_{\alpha'}\rangle$ *is invariant under the action of* X_i, X_{i+1}, *and* s_i, *and in the basis* $\{v_\alpha, v_{\alpha'}\}$, *these operators are given by the matrices*

$$\begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix}, \quad \begin{pmatrix} a_{i+1} & 0 \\ 0 & a_i \end{pmatrix}, \quad \textit{and} \quad \begin{pmatrix} \frac{1}{a_{i+1}-a_i} & 1-\frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i-a_{i+1}} \end{pmatrix},$$

respectively.

Proof. It follows immediately from the definitions of α and v_α that

$$X_i v_\alpha = a_i v_\alpha \quad \text{and} \quad X_{i+1} v_\alpha = a_{i+1} v_\alpha.$$

It also follows from (3.20) and (3.22) that $\langle v_\alpha, s_i v_\alpha\rangle$ is invariant under the action of X_i and X_{i+1}. It is also clearly invariant under the action of s_i.

Suppose that $s_i v_\alpha$ and v_α are linearly independent and $a_{i+1} = a_i \pm 1$. Then a direct computation shows that the only line stable under the action of the subalgebra $\mathcal{A}$ of $L(\mathfrak{S}_n)$ generated by s_i, X_i, and X_{i+1} is the line spanned by $s_i v_\alpha \mp v_\alpha$. This contradicts the fact that representations of $\mathcal{A}$ are completely reducible (since s_i, X_i, and X_{i+1} act by unitary operators). Thus, if $a_{i+1} = a_i \pm 1$, then the vectors $s_i v_\alpha$ and α are proportional, in which case (3.20) implies that

$$a_i s_i v_\alpha + v_\alpha = a_{i+1} s_i v_\alpha, \tag{3.24}$$

and so $s_i v_\alpha = \pm v_\alpha$. Conversely, if $s_i v_\alpha = \lambda v_\alpha$, then the fact that $s_i^2 = 1$ implies that $\lambda^2 = 1$, and so $\lambda = \pm 1$. Then (3.24) implies that $a_{i+1} = a_i \pm 1$. This proves (b).

Now suppose that $a_{i+1} \neq a_i \pm 1$. Then, by the above, we have

$$\dim\langle v_\alpha, s_i v_\alpha\rangle = 2.$$

By (3.20) and (3.22), we have

$$X_i s_i v_\alpha = -v_\alpha + a_{i+1} s_i v_\alpha,$$
$$X_{i+1} s_i v_\alpha = v_\alpha + a_i s_i v_\alpha.$$

Thus the actions of s_i, X_i, and X_{i+1} on $\langle v_\alpha, s_i v_\alpha\rangle$ are represented, with respect to the basis $\{v_\alpha, s_i v_\alpha\}$ by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} a_i & -1 \\ 0 & a_{i+1} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a_{i+1} & 1 \\ 0 & a_i \end{pmatrix},$$

respectively.

Now, we know that the actions of X_i and X_{i+1} are diagonalizable. This implies that $a_i \neq a_{i+1}$, proving (a). Then we check directly that

$$v' := s_i v_\alpha - \frac{1}{a_{i+1} - a_i} v_\alpha$$

is an eigenvector of X_i and X_{i+1} with eigenvalues a_{i+1} and a_i, respectively. In addition (3.21) implies that

$$X_j v' = a_j v', \quad \text{for all } j \neq i, i+1.$$

Thus

$$\alpha' := (a_1, a_2, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_n) \in \mathrm{Spec}(n),$$

and $v' = v_{\alpha'}$ is a vector of the Young basis. We leave it as an exercise (Exercise 3.3.2) to verify the formula for the action of s_i in the basis $\{v_\alpha, v_{\alpha'}\}$. □

Exercises.

3.3.2. In the notation of the proof of Proposition 3.3.2, prove that the action of s_i on the space $\langle v_\alpha, v_{\alpha'}\rangle$, with respect to the basis $\{v_\alpha, v_{\alpha'}\}$, is represented by the matrix

$$\begin{pmatrix} \frac{1}{a_{i+1}-a_i} & 1 - \frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i - a_{i+1}} \end{pmatrix}.$$

3.3.3 Spec(n) = Cont(n)

Let

$$\alpha = (a_1, a_2, \dots, a_n) \in \mathrm{Spec}(n).$$

We say that s_i is an *admissible transposition* for α if $a_{i+1} \neq a_i \pm 1$. In this case, by Proposition 3.3.2(c), we have

$$s_i\alpha = (a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_n) \in \mathrm{Spec}(n).$$

The Coxeter generators satisfy the relations (see Exercise 3.1.4)

$$s_i s_j = s_j s_i, \qquad 1 \le i, j \le n-1,\ |i-j| > 1, \tag{3.25}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad 1 \le i \le n-2. \tag{3.26}$$

Lemma 3.3.3. *Let $\alpha = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n$. If $a_i = a_{i+2} = a_{i+1} \pm 1$ for some $i \in \{1, 2, \dots, n-2\}$, then $\alpha \notin \mathrm{Spec}(n)$.*

Proof. Suppose, towards a contradiction, that $a_i = a_{i+2} = a_{i+1} - 1$ and $\alpha \in \mathrm{Spec}(n)$. Then, by Proposition 3.3.2(b), we have

$$s_i v_\alpha = v_\alpha \quad \text{and} \quad s_{i+1} v_\alpha = -v_\alpha.$$

Then, by (3.26), we have

$$v_\alpha = s_{i+1} s_i s_{i+1} v_\alpha = s_i s_{i+1} s_i v_\alpha = -v_\alpha,$$

which contradicts the fact that $v_\alpha \neq 0$. The proof of the case that $a_i = a_{i+2} = a_{i+1} + 1$ is similar. □

Lemma 3.3.4. *(a) For all $(a_1, a_2, \dots, a_n) \in \mathrm{Spec}(n)$, we have $a_1 = 0$.*

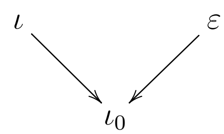
(b) If $(a_1, a_2, \ldots, a_n) \in \mathrm{Spec}(n)$*, then* $(a_1, a_2, \ldots, a_{n-1}) \in \mathrm{Spec}(n-1)$*.*

(c) We have $\mathrm{Spec}(2) = \{(0,1), (0,-1)\}$*.*

Proof. (a) This follows immediately from the fact that $X_1 = 0$.

(b) This follows from the fact that $X_1, X_2, \ldots, X_{n-1} \in L(\mathfrak{S}_{n-1})$ and $X_j v_\alpha = a_j v_\alpha$ for all $1 \le j \le n-1$.

(c) The group $\mathfrak{S}_2$ has two irreducible representations: the trivial representation ι and the sign representation ε (see Example 1.1.5). The branching graph of $\mathfrak{S}_1 \le \mathfrak{S}_2$ is



where ι_0 is the trivial representation of $\mathfrak{S}_1$. Since $X_2 = (1,2)$, we have

$$X_2 v = \begin{cases} v & \text{if } v \in V_\iota, \\ -v & \text{if } v \in V_\varepsilon. \end{cases}$$ □

Lemma 3.3.5. *(a) For all* $n \ge 1$*, we have* $\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$*.*

(b) If $\alpha \in \mathrm{Spec}(n)$*,* $\beta \in \mathrm{Cont}(n)$*, and* $\alpha \approx \beta$*, then* $\beta \in \mathrm{Spec}(n)$ *and* $\alpha \sim \beta$*.*

Proof. We prove (a) by induction on n. The case $n = 1$ is trivial, while the case $n = 2$ follows from Lemma 3.3.4(c) and (3.5).

Suppose that $\mathrm{Spec}(n-1) \subseteq \mathrm{Cont}(n-1)$, and let

$$\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{Spec}(n).$$

By Lemma 3.3.4(a), we have $a_1 = 0$, which corresponds to (a) of Definition 3.1.7. By Lemma 3.3.4(b) and our induction hypothesis, to prove that $\alpha \in \mathrm{Spec}(n)$, it suffices to prove that conditions (b) and (c) of Definition 3.1.7 are satisfied for $j = n$.

First suppose, towards a contradiction, that α does not satisfy Definition 3.1.7(b). In other words, we suppose that

$$\{a_n - 1, a_n + 1\} \cap \{a_1, a_2, \ldots, a_{n-1}\} = \varnothing. \tag{3.27}$$

By Proposition 3.3.2(c), the transposition $(n-1, n)$ is admissible for α, that is

$$(a_1, a_2, \ldots, a_{n-2}, a_n, a_{n-1}) \in \mathrm{Spec}(n).$$

Then, by Lemma 3.3.4(b) and our induction hypothesis, we have

$$(a_1, a_2, \ldots, a_{n-2}, a_n) \in \mathrm{Spec}(n-1) \subseteq \mathrm{Cont}(n-1).$$

By (3.27), we have

$$\{a_n - 1, a_n + 1\} \cap \{a_1, a_2, \ldots, a_{n-2}\} = \varnothing.$$

But this contradicts Definition 3.1.7(b) for $\mathrm{Cont}(n-1)$. This completes the proof that (b) of Definition 3.1.7 is satisfied for $\mathrm{Cont}(n)$.

Now suppose, towards a contradiction, that α does not satisfy Definition 3.1.7(c) for $j = n$. In other words, we suppose that $a_i = a_n = a$ for some $i < n$ and, for instance,

$$a - 1 \notin \{a_{i+1}, a_{i+2}, \dots, a_{n-1}\}.$$

(The case $a + 1 \notin \{a_{i+1}, a_{i+2}, \dots, a_{n-1}\}$ is similar and will be omitted.) We choose i to be maximal with the above properties, so that we also have

$$a \notin \{a_{i+1}, a_{i+2}, \dots, a_{n-1}\}. \tag{3.28}$$

By the inductive hypothesis, we have $(a_1, a_2, \dots, a_{n-1}) \in \mathrm{Cont}(n-1)$. Thus $a+1$ may appear at most once in $a_{i+1}, a_{i+2}, \dots, a_{n-1}$ since, if it appeared more than once, Definition 3.1.7(c) for $\mathrm{Cont}(n-1)$ would imply that a appears between two occurrences of $a+1$ in $a_{i+1}, a_{i+2}, \dots, a_{n-1}$, contradicting (3.28). Suppose

$$a + 1 \notin \{a_{i+1}, a_{i+2}, \dots, a_{n-1}\}.$$

Then

$$(a_i, a_{i+1}, \dots, a_n) = (a, *, \dots, *, a)$$

where all the entries $*$ are different from a, $a+1$, and $a-1$. Then, by a sequence of $n-i-1$ admissible transpositions, we get

$$\alpha \sim (\dots, a, a, \dots) \in \mathrm{Spec}(n),$$

which contradicts Proposition 3.3.2(a).

Similarly, if

$$a + 1 \in \{a_{i+1}, a_{i+2}, \dots, a_{n-1}\},$$

then

$$(a_i, a_{i+1}, \dots, a_n) = (a, *, \dots, *, a+1, *, \dots, *, a),$$

where, as before, each $*$ represents a number not equal to a, $a+1$, or $a-1$. Then, by a sequence of admissible transpositions, we get

$$\alpha \sim (\dots, a, a+1, a, \dots) \in \mathrm{Spec}(n),$$

which contradicts Lemma 3.3.3. This completes the proof that (c) of Definition 3.1.7 is satisfied for $\mathrm{Cont}(n)$, completing the proof of part (a) of the current lemma.

Part (b) of the current lemma is an immediate consequence of part (a), Corollary 3.1.9, and Proposition 3.3.2(c). □

Theorem 3.3.6. *(a) We have* $\mathrm{Spec}(n) = \mathrm{Cont}(n)$.

(b) The equivalence relations $\sim$ *and* $\approx$ *are the same.*

(c) The Young graph $\mathbb{Y}$ *is isomorphic to the branching graph of the multiplicity-free chain* $\mathfrak{S}_1 \le \mathfrak{S}_2 \le \cdots \le \mathfrak{S}_n \le \mathfrak{S}_{n+1} \le \cdots$.

Proof. First note that

$$\begin{aligned} |\operatorname{Cont}(n)/\approx| &= \text{number of partitions of } n && \text{(Corollary 3.1.10)}\\ &= \text{number of conjugacy classes of } n && \text{(Proposition 3.1.2)}\\ &= \left|\widehat{\mathfrak{S}_n}\right| && \text{(Proposition 1.3.16)}\\ &= |\operatorname{Spec}(n)/\sim| && \text{(by (3.19)).} \end{aligned}$$

By Lemma 3.3.5, each equivalence class in $\operatorname{Cont}(n)/\approx$ is either disjoint from $\operatorname{Spec}(n)$ or is contained in a single equivalence class in $\operatorname{Spec}(n) \sim$. In particular, the partition of $\operatorname{Spec}(n)$ induced by $\approx$ is finer than the partition of $\operatorname{Spec}(n)$ induced by $\sim$. Therefore

$$|\operatorname{Spec}(n)/\sim| \le |\operatorname{Spec}(n)/\approx| \le |\operatorname{Cont}(n)/\approx| = |\operatorname{Spec}(n)/\sim|.$$

It follows that the two inequalities above must be equality, proving (a) and (b).

As explained at the beginning of Section 3.3.2, $\operatorname{Spec}(n)$ parameterizes the paths in the branching graph. By (3.8), $\operatorname{Cont}(n)$ parameterizes the paths in $\mathbb{Y}$. Thus, by part (a), we have a bijective correspondence between the paths in $\mathbb{Y}$ and the paths in the branching graph. By Proposition 3.1.11 and the definition of $\sim$, this yields a bijective correspondence between the vertices of these graphs. This correspondence is clearly a graph isomorphism. □

It follows from Theorem 3.3.6 that we have a natural correspondence between $\widehat{\mathfrak{S}_n}$ and the n-th level of the branching graph $\mathbb{Y}$, that is, the set of all partitions of n.

Definition 3.3.7 (The irreducible representations S^λ)**.** Given a partition $\lambda \vdash n$, we define S^λ to be the irreducible representation of $\mathfrak{S}_n$ spanned by the vectors v_α, with $\alpha \in \operatorname{Spec}(n) = \operatorname{Cont}(n)$ corresponding to a standard tableau of shape λ.

Proposition 3.3.8. *We have* $\dim S^\lambda = |\operatorname{Tab}(\lambda)|$. *In other words, the dimension of* S^λ *is equal to the number of standard tableaux of shape* λ.

Proof. This follows immediately from Definition 3.3.7. □

Proposition 3.3.2 will allow us to give explicit formulas for the action of the Coxeter generators (and hence any element of $\mathfrak{S}_n$) on the irreducible representations S^λ. See Theorems 3.4.2 and 3.4.4.

Corollary 3.3.9. *Suppose* $0 \le k < n$, $\lambda \vdash n$, *and* $\mu \vdash k$. *The multiplicity of* S^μ *in* $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda$ *is equal to the number of paths in* $\mathbb{Y}$ *from* λ *to* μ.

Proof. We have

$$\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda = \operatorname{Res}^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} \operatorname{Res}^{\mathfrak{S}_{k+2}}_{\mathfrak{S}_{k+1}} \cdots \operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\lambda.$$

At each step of the right side, the decomposition is multiplicity free and according to the branching graph $\mathbb{Y}$. Thus, the multiplicity of S^μ in $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_k} S^\lambda$ is equal to the number of paths in $\mathbb{Y}$ that start at λ and end at μ. □

Corollary 3.3.10 (Branching rule)**.** *For $\lambda \vdash n$,*

$$\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\lambda = \bigoplus_{\substack{\mu \vdash n-1:\\ \lambda \to \mu}} S^\mu. \tag{3.29}$$

The sum above runs over all partitions $\mu \vdash n-1$ obtained from λ by removing a single box. Moreover, for all $\mu \vdash n-1$, we have

$$\operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\mu = \bigoplus_{\substack{\lambda \vdash n:\\ \lambda \to \mu}} S^\lambda. \tag{3.30}$$

Proof. Corollary 3.3.9 immediately implies (3.29). Now suppose $\mu \vdash n-1$ and $\lambda \vdash n$. Then we have

$$\begin{aligned}
&\dim \operatorname{Hom}_{\mathfrak{S}_n}\left(S^\lambda, \operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\mu\right)\\
&\qquad = \dim \operatorname{Hom}_{\mathfrak{S}_{n-1}}\left(\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\lambda, S^\mu\right) && \text{(Frobenius reciprocity (Theorem 1.6.6))}\\
&\qquad = \dim \operatorname{Hom}_{\mathfrak{S}_{n-1}}\left(\bigoplus_{\substack{\nu \vdash n-1:\\ \lambda \to \nu}} S^\nu, S^\mu\right) && \text{(by (3.29))}\\
&\qquad = \begin{cases} 1, & \text{if } \lambda \to \mu,\\ 0, & \text{otherwise.}\end{cases} && \text{(Schur's lemma (Corollary 1.2.2)).}
\end{aligned}$$

Thus (3.30) follows from Corollary 1.2.6. □

We have defined two notions of admissible transposition, one for $\operatorname{Spec}(n)$, and one for $\operatorname{Cont}(n)$ (coming from the notion of admissible transposition for tableaux). The following result states that these coincide.

Lemma 3.3.11. *Suppose $T \in \operatorname{Tab}(n)$ is a standard tableau with content $\alpha = C(T) = (a_1, a_2, \dots, a_n) \in \operatorname{Spec}(n)$. For $1 \le i \le n-1$, the simple transposition s_i is admissible for T if and only if $a_{i+1} \ne a_i \pm 1$.*

Proof. We have

$$\begin{aligned}
a_{i+1} = a_i \pm 1 \iff &\text{ the box labelled } i+1 \text{ is immediately to the right of}\\
&\quad \text{or immediately below the box labelled } i \text{ in } T\\
\iff &\ i \text{ and } i+1 \text{ belong to the same row or column of } T\\
\iff &\ s_i \text{ is not admissible for } T.
\end{aligned}$$

□

Exercises.

3.3.3. Fix $n \geq 2$. Determine the partitions $\lambda, \mu \vdash n$ such that S^λ is the trivial representation and S^μ is the sign representation (see Example 1.1.5). *Hint*: Use Proposition 3.3.2.

3.3.4. Fix $n \geq 2$. Prove that $\mathfrak{S}_n$ has exactly two inequivalent one-dimensional irreducible representations.

3.4 The irreducible representations of $\mathfrak{S}_n$

In this section we will deduce explicit descriptions of the irreducible representations. In particular, we will compute matrix coefficients for the simple transpositions. We will also derive a formula for the primitive idempotents in terms of the YJM elements. Finally, we state a theorem of Jucys and Murphy relating the centre of the group algebra $L(\mathfrak{S}_n)$ and the YJM elements.

3.4.1 Young's seminormal form

For a partition $\lambda \vdash n$, recall the tableau T^λ defined in Section 3.1.4. Furthermore, recall that for any $T \in \mathrm{Tab}(\lambda)$, there exists a unique permutation $\pi_T \in \mathfrak{S}_n$ such that $\pi_T T = T^\lambda$.

Recall that the Young vector v_T associated to a tableau T is defined up to a scalar factor (of norm one if the Young vectors are normalized).

Proposition 3.4.1. *It is possible to choose the vectors v_T, $T \in \mathrm{Tab}(n)$, such that, for every $T \in \mathrm{Tab}(\lambda)$, $\lambda \vdash n$, we have*

$$\pi_T^{-1} v_{T^\lambda} = v_T + \sum_{\substack{R \in \mathrm{Tab}(\lambda):\\ \ell(\pi_R) < \ell(\pi_T)}} \gamma_R v_R,$$

for some $\gamma_R \in \mathbb{C}$.

Proof. We prove the result by induction on $\ell(\pi_T)$. At each step in the induction, we will choose the vectors v_T for all T with $\ell(v_T) = \ell$.

If $\ell(\pi_T) = 1$, then π_T is an admissible transposition for T^λ. In this case, the result follows from Proposition 3.3.2 (and Lemma 3.3.11). In particular, we choose v_T to be the $v_{\alpha'}$ appearing in (3.23).

Now suppose $\pi_T = s_{i_1} s_{i_2} \cdots s_{i_{\ell-1}} s_j$ is the standard decomposition of π_T into a product of admissible transpositions (see Remark 3.1.6). Then $\pi_T = \pi_{T_1} s_j$, where $T_1 = s_j T$ is a standard tableau and $\ell(\pi_{T_1}) = \ell(\pi_T) - 1$.

By the inductive hypothesis, we have

$$\pi_{T_1}^{-1} v_{T^\lambda} = v_{T_1} + \sum_{\substack{R \in \mathrm{Tab}(\lambda):\\ \ell(\pi_R) < \ell(\pi_{T_1})}} \gamma_R^{(1)} v_R, \tag{3.31}$$

for some $\gamma_R^{(1)} \in \mathbb{C}$. Since $T = s_j T_1$, then as in (3.23) we can choose v_T such that

$$s_j v_{T_1} = v_T + \frac{1}{a_{j+1} - a_j} v_{T_1}, \tag{3.32}$$

where $(a_1, a_2, \ldots, a_n) = C(T_1)$ is the content of T_1. Then we have

$$\begin{aligned}\pi_T^{-1} v_{T^\lambda} = s_j \pi_{T_1}^{-1} v_{T^\lambda} &\overset{(3.31)}{=} s_j v_{T_1} + \sum_{\substack{R \in \mathrm{Tab}(\lambda): \\ \ell(\pi_R) < \ell(\pi_{T_1})}} \gamma_R^{(1)} s_j v_R \\ &\overset{(3.32)}{=} v_T + \frac{1}{a_{j+1} - a_j} v_{T_1} + \sum_{\substack{R \in \mathrm{Tab}(\lambda): \\ \ell(\pi_R) < \ell(\pi_{T_1})}} \gamma_R^{(1)} s_j v_R\end{aligned}$$

Then the result follows by using Proposition 3.3.2 to compute the terms $s_j v_R$. □

Theorem 3.4.2 (Young's seminormal form)**.** *Choose the vectors of the Young basis according to Proposition 3.4.1. If $T \in \mathrm{Tab}(\lambda)$ has content $C(T) = (a_1, a_2, \ldots, a_n)$, then the simple transposition s_j acts on v_T as follows:*

(a) If $a_{j+1} = a_j \pm 1$, then $s_j v_T = \pm v_T$.

(b) If $a_{j+1} \neq a_j \pm 1$, then, setting $T' = s_j T$, we have

$$s_j v_T = \begin{cases} \frac{1}{a_{j+1} - a_j} v_T + v_{T'} & \text{if } \ell(\pi_{T'}) > \ell(\pi_T), \\ \frac{1}{a_{j+1} - a_j} v_T + \left(1 - \frac{1}{(a_{j+1} - a_j)^2}\right) v_{T'} & \text{if } \ell(\pi_{T'}) < \ell(\pi_T). \end{cases} \tag{3.33}$$

Proof. Part (a) follows immediately from Proposition 3.3.2(b).

Suppose that $\ell(\pi_{T'}) > \ell(\pi_T)$. By Proposition 3.3.2(c), we have

$$s_j v_T = c v_{T'} + \frac{1}{a_{i+1} - a_i} v_T$$

for some nonzero $c \in \mathbb{C}$ (since (3.23) holds up for $v_{\alpha'}$ up to a scalar factor).

Since $\pi_{T'} = \pi_T s_j$, we have, by Proposition 3.4.1

$$v_{T'} + \sum_{\substack{R' \in \mathrm{Tab}(\lambda): \\ \ell(\pi_{R'}) < \ell(\pi_{T'})}} \gamma'_{R'} v_{R'} = \pi_{T'}^{-1} v_{T^\lambda} = s_j \pi_T^{-1} v_{T^\lambda} = s_j v_T + \sum_{\substack{R \in \mathrm{Tab}(\lambda): \\ \ell(\pi_R) < \ell(\pi_T)}} \gamma_R s_j v_R$$

Thus $c = 1$.

The case $\ell(\pi_{T'}) < \ell(\pi_T)$ is similar, starting from $\pi_T = \pi_{T'} s_j$ and applying Proposition 3.3.2 with $\alpha = C(T')$. (See Exercise 3.4.1.) □

Corollary 3.4.3. *In the orthogonal bases of Proposition 3.4.1 and Theorem 3.4.2 the matrix coefficients of the irreducible representations of $\mathfrak{S}_n$ are* rational *numbers. In particular, the coefficients γ_R in Proposition 3.4.1 are rational numbers.*

Exercises.

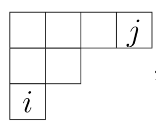
3.4.1. Complete the proof of Theorem 3.4.2(b) by treating the case $\ell(\pi_{T'}) < \ell(\pi_T)$.

3.4.2 Young's orthogonal form

The orthogonal bases of Proposition 3.4.1 and Theorem 3.4.2 do not consist of unit vectors in general. Given an arbitrary invariant scalar product $\| \ \|$ on S^λ that makes it a unitary representation of $\mathfrak{S}_n$ (see Lemma 1.1.1) we can, of course, normalize the basis. If $\lambda \vdash n$ and $\{v_T : T \in \mathrm{Tab}(\lambda)\}$ is the basis as in Proposition 3.4.1 and Theorem 3.4.2, we define

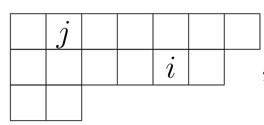
$$w_T = \frac{v_T}{\|v_T\|}, \quad T \in \mathrm{Tab}(\lambda).$$

Let T be a standard tableau and let $C(T) = (a_1, a_2, \dots, a_n)$ be its content. For $i, j \in \{1, 2, \dots, n\}$, the *axial distance* from j to i in T is the integer $a_j - a_i$. Geometrically, we move from j to i in the tableau T, counting each step left or downwards as $+1$ and each step right or upwards as -1. The resulting integer is $a_j - a_i$ (and is independent of the path chosen). For example, if we have



,

then the axial distance from j to i is $a_j - a_i = 5$. On the other hand, if we have



,

then the axial distance from j to i is $a_j - a_i = -2$.

Theorem 3.4.4 (Young's orthogonal form)**.** *In the orthonormal basis $\{w_T : T \in \mathrm{Tab}(n)\}$ we have*

$$s_j w_T = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}} w_{s_j T}, \tag{3.34}$$

where r is the axial distance from $j+1$ to j in T. In particular, if $a_{j+1} = a_j \pm 1$, we have $r = \pm 1$ and $s_j w_T = \pm w_T$.

Proof. Let $T' = s_j T$ and suppose $\ell(\pi_{T'}) > \ell(\pi_T)$. By Theorem 3.4.2, we have

$$\begin{aligned} \|v_{T'}\|^2 &= \left\| s_j v_T - \frac{1}{r} v_T \right\|^2 \\ &= \|s_j v_T\|^2 - \frac{1}{r}\langle s_j v_T, v_T\rangle - \frac{1}{r}\langle v_T, s_j v_T\rangle + \frac{1}{r^2}\|v_T\|^2 \\ &= \|v_T\|^2 - \frac{1}{r}\left\langle \frac{1}{r} v_T + v_{T'}, v_T \right\rangle - \frac{1}{r}\left\langle v_T, \frac{1}{r} v_T + v_{T'} \right\rangle + \frac{1}{r^2}\|v_T\|^2 \end{aligned}$$

$$= \left(1 - \frac{1}{r^2}\right) \|v_T\|^2,$$

where we have used the fact that $v_T \perp v_{T'}$. Thus we have

$$w_T = \frac{v_T}{\|v_T\|} \qquad \text{and} \qquad w_{T'} = \frac{v_{T'}}{\|v_{T'}\|} = \frac{v_{T'}}{\sqrt{1 - \frac{1}{r^2}}\,\|v_T\|}.$$

In this basis, the first case in (3.33) becomes (3.34).

The proof in the case $\ell(\pi_{T'}) < \ell(\pi_T)$ is similar (Exercise 3.4.2). □

It follows from Theorem 3.4.4 that

$$s_j w_T = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}}\, w_{s_j T}, \qquad s_j w_{s_j T} = -\frac{1}{r} w_{s_j T} + \sqrt{1 - \frac{1}{r^2}}\, w_T, \tag{3.35}$$

where r is the axial distance from $j+1$ to j. Thus, in the bases $\{w_T, w_{s_j T}\}$, the action of s_j is given by the orthogonal matrix

$$\begin{pmatrix} \frac{1}{r} & \sqrt{1 - \frac{1}{r^2}} \\ \sqrt{1 - \frac{1}{r^2}} & -\frac{1}{r} \end{pmatrix}.$$

Example 3.4.5. The only standard tableau of shape (n) is

$$T = T^{(n)} = \boxed{1}\boxed{2}\boxed{}\boxed{}\boxed{}\boxed{n}\,.$$

Its content is $C(T) = (0, 1, 2, \dots, n-1)$. Thus $a_{j+1} = a_j + 1$ for all $1 \le j \le n-1$. By (3.34),

$$s_j w_T = w_T, \quad \text{for all } 1 \le j \le n-1.$$

Hence $S^{(n)}$ is the trivial representation of $\mathfrak{S}_n$.

Example 3.4.6. The only standard tableau of shape $(1, 1, \dots, 1)$ is

$$T = T^{(1,1,\dots,1)} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \\ \hline \\ \hline \\ \hline n \\ \hline \end{array}\,.$$

Its content is $C(T) = (0, -1, -2, \dots, -n+1)$. Thus $a_{j+1} = a_j - 1$ for all $1 \le j \le n-1$. By (3.34),

$$s_j w_T = -w_T, \quad \text{for all } 1 \le j \le n-1.$$

Hence $S^{(1,1,\dots,1)}$ is the sign representation of $\mathfrak{S}_n$.

Example 3.4.7. Consider the representation $S^{(n-1,1)}$. The standard tableaux of shape $(n-1,1)$ are

$$T_j = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & & & j-1 & j+1 & & & n \\ \hline j \\ \cline{1-1}\end{array}, \quad 2 \leq j \leq n.$$

For $2 \leq j \leq n$, we have

$$C(T_j) = (0, 1, \dots, j-2, -1, j-1, j, \dots, n-2), \tag{3.36}$$

where the entry -1 is in the j-th position. Let $w_j = w_{T_j}$ for $2 \leq j \leq n$. Then the Young orthogonal form becomes

$$s_j w_j = \frac{1}{j} w_j + \sqrt{1 - \frac{1}{j^2}}\, w_{j+1}, \tag{3.37}$$

$$s_{j-1} w_j = -\frac{1}{j-1} w_j + \sqrt{1 - \frac{1}{(j-1)^2}}\, w_{j-1}, \tag{3.38}$$

$$s_k w_j = w_j, \quad k \neq j-1, j. \tag{3.39}$$

The branching rule gives

$$\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^{(n-1,1)} = S^{(n-1)} \oplus S^{(n-2,1)}.$$

We claim that $S^{(n-1,1)}$ is isomorphic to the representation V_1 of Example 1.4.5. We prove this claim by giving an explicit isomorphism. Let $X = \{1, 2, \dots, n\}$ and recall that

$$V_1 = \left\{ f \in L(X) : \sum_{j=1}^{n} f(j) = 0 \right\}.$$

For $2 \leq j \leq n$, define

$$\tilde{w}_j = \frac{1}{\sqrt{j(j-1)}} \mathbf{1}_{j-1} - \sqrt{\frac{j-1}{j}}\, \delta_j, \tag{3.40}$$

where δ_j is the Dirac function at j, and $\mathbf{1}_j = \delta_1 + \delta_2 + \cdots + \delta_j$.

One can check (Exercise 3.4.3) that $\{\tilde{w}_j : 2 \leq j \leq n\}$ is an orthonormal basis for V_1. Now,

$$\begin{aligned} \frac{1}{j}\tilde{w}_j + \sqrt{1 - \frac{1}{j^2}}\,\tilde{w}_{j+1} &= \frac{1}{j\sqrt{j(j-1)}} \mathbf{1}_{j-1} - \frac{1}{j}\sqrt{\frac{j-1}{j}}\,\delta_j + \frac{1}{j}\sqrt{\frac{j-1}{j}}\,\mathbf{1}_j - \sqrt{\frac{j-1}{j}}\,\delta_{j+1} \\ &= \frac{1}{\sqrt{j(j-1)}} \mathbf{1}_{j-1} - \sqrt{\frac{j-1}{j}}\,\delta_{j+1} = s_j \tilde{w}_j. \end{aligned}$$

Thus, the vectors $\tilde{w}_j$ satisfy (3.37). The proof that they satisfy (3.38) is similar. That fact that they satisfy (3.39) is straightforward. Therefore we have an isomorphism of representations

$$V_1 \to S^{(n-1,1)}, \quad \tilde{w}_j \mapsto w_j, \quad 2 \leq j \leq n.$$

Exercises.

3.4.2. Complete the proof of Theorem 3.4.4 in the case $\ell(\pi_{T'}) < \ell(\pi_T)$.

3.4.3. We adopt the notation of Example 3.4.7.

(a) Verify that $\{\tilde{w}_j : 2 \leq j \leq n\}$ is an orthonormal basis for V_1.

(b) Verify that the vectors $\tilde{w}_j$ satisfy (3.38).

3.4.4. Prove that $\tilde{w}_j$, as defined in (3.40), corresponds to the path

$$(n-1,1) \to (n-2,1) \to \cdots \to (j,1) \to (j-1,1) \to (j-1) \to (j-2) \to \cdots \to (2) \to (1)$$

in the Young graph by examining the action of $\mathfrak{S}_n \geq \mathfrak{S}_{n-1} \geq \cdots \geq \mathfrak{S}_1$. This gives another way of identifying $\tilde{w}_j$ with w_j.

3.4.5. Define $\tilde{w}_j$ as in (3.40). Show, by direct computation, that

$$X_j \tilde{w}_j = \begin{cases} (i-1)\tilde{w}_j & \text{for } i < j, \\ -\tilde{w}_j & \text{for } i = j, \\ (i-2)\tilde{w}_j & \text{for } i > j. \end{cases}$$

In other words, prove that $\alpha(T_j) = (0, 1, 2, \ldots, j-2, -1, j-1, j, \ldots, n-2) \in \mathrm{Spec}(n)$. Compare with (3.36).

3.4.3 The Young seminormal units

In this section we will deduce an expression, in terms of YJM elements, for the primitive idempotents of $L(\mathfrak{S}_n)$ corresponding to the Gelfand–Tsetlin bases for the irreducible representations. (See Proposition 1.5.8.)

For $\lambda \vdash n$, let

$$d_\lambda = \dim S^\lambda.$$

For each $T = \mathrm{Tab}(n)$ of shape λ, the primitive idempotent in $L(\mathfrak{S}_n)$ corresponding to the Gelfand–Tsetlin vector w_T is given by (see (1.46))

$$e_T(\pi) = \frac{d_\lambda}{n!} \langle \pi w_T, w_T \rangle_{S^\lambda}, \quad \pi \in \mathfrak{S}_n. \tag{3.41}$$

Following the notation of this chapter, we will identify e_T with the formal sum

$$\sum_{\pi \in \mathfrak{S}_n} e_T(\pi)\pi.$$

For $S \in \mathrm{Tab}(n)$, we let

$$e_T w_S = \sum_{\pi \in \mathfrak{S}_n} e_T(\pi)\pi w_s$$

denote the action of e_T (more precisely, of its Fourier transform) on w_S. Furthermore, $e_T e_S$ denotes the convolution of e_T and e_S.

By Proposition 1.5.8, for all $S, T \in \mathrm{Tab}(n)$ we have

$$e_T e_S = \delta_{T,S} e_T \qquad \text{and} \tag{3.42}$$

$$e_T w_S = \delta_{T,S} w_T. \tag{3.43}$$

Note that (3.43) uniquely characterizes e_T among elements of $L(\mathfrak{S}_n)$. Also, it follows from (3.43) and Theorem 2.2.2 that

$$\{e_T : T \in \mathrm{Tab}(n)\}$$

is a basis of the Gelfand–Tsetlin algebra $\mathrm{GZ}(n)$. The elements e_T, $T \in \mathrm{Tab}(n)$, are called *Young seminormal units*.

For $T \in \mathrm{Tab}(n)$, let $\overline{T} \in \mathrm{Tab}(n-1)$ denote the tableau obtained from T by removing the box labelled n. We also denote by $a_T(j)$ the j-th component of $C(T)$; in other words,

$$C(T) = (a_T(1), a_T(2), \ldots, a_T(n)).$$

It follows that

$$X_k w_T = a_T(k) w_T, \quad \text{for all } T \in \mathrm{Tab}(n),\ 1 \le k \le n.$$

The following theorem gives a recursive formula for the Young seminormal units in terms of the YJM elements.

Theorem 3.4.8. *We have $e_T = 1$ for the unique $T \in \mathrm{Tab}(1)$. For $n \ge 2$ and $T \in \mathrm{Tab}(n)$, we have*

$$e_T = e_{\overline{T}} \prod_{\substack{S \in \mathrm{Tab}(n): \\ \overline{S} = \overline{T},\, S \neq T}} \frac{X_n - a_S(n)}{a_T(n) - a_S(n)}. \tag{3.44}$$

Proof. Let $\tilde{e}_T$ denote the element of $L(\mathfrak{S}_n)$ recursively defined by the right side of (3.44). We will show that $\tilde{e}_T w_S = \delta_{T,S} w_S$ for all $S \in \mathrm{Tab}(n)$. By the characterizing property (3.43), this will imply that $\tilde{e}_T = e_T$.

We proceed by induction on n. The result is clearly true for $n = 1$. Thus, we assume $n \ge 2$ and that the result is true for $n-1$.

Suppose $\overline{S} \neq \overline{T}$. Then

$$\tilde{e}_{\overline{T}} w_S = \tilde{e}_{\overline{T}} w_{\overline{S}} = 0.$$

The first equality above follows from the fact that $\tilde{e}_{\overline{T}} \in L(\mathfrak{S}_{n-1})$, and so, to compute the action of $\tilde{e}_{\overline{T}}$ on w_S, we first restrict the irreducible representation containing w_S to $\mathfrak{S}_{n-1}$, obtaining the vector $w_{\overline{S}}$. The second equality above follows from the induction hypothesis.

Now suppose $\overline{S} = \overline{T}$, but $S \neq T$. Then

$$X_n w_S = a_S(n) w_S,$$

and so $\tilde{e}_T w_S = 0$ since the factor $X_n - a_S(n)$ in the right side of (3.44) acts as zero on w_S.

Finally, note that $X_n w_T = a_T(n) w_T$, and so

$$\frac{X_n - a_S(n)}{a_T(n) - a_S(n)} w_T = w_T$$

for all $S \in \mathrm{Tab}(n)$ such that $\overline{S} = \overline{T}$ and $S \neq T$. Hence

$$\begin{aligned}\tilde{e}_T w_T &= \tilde{e}_{\overline{T}} \cdot \prod_{\substack{S \in \mathrm{Tab}(n):\\ \overline{S}=\overline{T},\ S\neq T}} \frac{X_n - a_S(n)}{a_T(n) - a_S(n)} w_T \\ &= \tilde{e}_{\overline{T}} w_T \\ &= \tilde{e}_{\overline{T}} w_{\overline{T}} \\ &= w_T,\end{aligned}$$

where the final two equalities follow from restriction to $\mathfrak{S}_{n-1}$ and the induction hypothesis. □

Corollary 3.4.9. *For $1 \leq k \leq n$, we have*

$$X_k = \sum_{T \in \mathrm{Tab}(n)} a_T(k) e_T.$$

Proof. This follows immediately from Theorem 3.4.8 and Proposition 1.5.8(d). □

Exercises.

3.4.6 ([CSST10, Ex. 3.4.13]). (a) Let T be the unique standard tableau of shape n (see Example 3.4.5). Show that

$$e_T = \frac{1}{n!} \prod_{j=1}^{n} (1 + X_j).$$

Prove also that $e_T = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \pi$ in two ways: (i) by means of the representation theory of $\mathfrak{S}_n$, and (ii) as an algebraic identity in $\mathfrak{S}_n$. *Hint*: If $\pi \in \mathfrak{S}_n$, then there exists a unique $\sigma \in \mathfrak{S}_{n-1}$ such that $\sigma \in \mathfrak{S}_{n-1}$ and $j \in \{1, 2, \dots, n-1\}$ such that $\pi = \sigma(j \to n \to j)$.

(b) Let T be the unique standard of shape $(1^n) = (1, \dots, 1) \vdash n$ (see Example 3.4.6). Show that

$$e_T = \frac{1}{n!} \prod_{j=1}^{n} (1 - X_j).$$

As in (a), give two proofs of the fact that $e_T = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} (-1)^{\ell(\pi)} \pi$.

(c) Let T_j be the standard tableau of Example 3.4.7. Show that

$$e_{T_j} = -\frac{(j-2)!}{n!(n-2)!} \left(\prod_{i=1}^{j-1} (X_i + 1) \right) \cdot (X_j - j + 1) \cdot \left(\prod_{i=j+1}^{n} X_i(X_i + 2) \right).$$

3.4.4 The Theorem of Jucys and Murphy

Consider the polynomial algebra $\mathbb{C}[y_1, \dots, y_m]$. The symmetric group $\mathfrak{S}_m$ acts on $\mathbb{C}[y_1, \dots, y_m]$ by permuting the indeterminates $y_1, \dots, y_m$. In other words, for $\pi \in \mathfrak{S}_m$, we have a unique algebra isomorphism

$$\mathbb{C}[y_1, \dots, y_m] \to \mathbb{C}[y_1, \dots, y_m], \quad f \mapsto \pi \cdot f,$$

determined by $\pi \cdot y_i = y_{\pi(i)}$.

The subalgebra

$$\mathbb{C}[y_1, \dots, y_m]^{\mathfrak{S}_m} = \{f \in \mathbb{C}[y_1, \dots, y_m] : \pi \cdot f = f \text{ for all } \pi \in \mathfrak{S}_m\} \subseteq \mathbb{C}[y_1, \dots, y_m]$$

is called the algebra of *symmetric polynomials*. So an element $f \in \mathbb{C}[y_1, \dots, y_m]$ is a symmetric polynomial if and only if it is left unchanged by any permutation of the indeterminates $y_1, \dots, y_m$.

Recall that $Z(n) = Z(L(\mathfrak{S}_n))$ is the center of the group algebra $L(\mathfrak{S}_n)$. We say an element $f \in Z(n)$ is a *symmetric polynomial in the YJM elements* if

$$f = p(X_2, \dots, X_n) \text{ for some } p \in \mathbb{C}[y_1, \dots, y_{n-1}]^{\mathfrak{S}_{n-1}}.$$

(In fact, recalling that $X_1 = 0$, one can show that this is equivalent to requiring that $f = p(X_1, X_2, \dots, X_n)$ for some $p \in \mathbb{C}[y_1, \dots, y_n]^{\mathfrak{S}_n}$.)

Theorem 3.4.10 (Theorem of Jucys and Murphy)**.** *The center $Z(n)$ of the group algebra of the symmetric group $\mathfrak{S}_n$ is precisely the algebra of all symmetric polynomials in the YJM elements $X_2, X_3, \dots, X_n$.*

Proof. Due to lack of time, we will not prove this theorem in this course. A proof can be found in [CSST10, Th. 4.4.5] or in [Mur83, Th. 1.9]. □

Chapter 4

Further directions

In this final chapter, we briefly touch on some more advanced topics related to the representation theory of the symmetric group and related algebras. We will omit the proofs of most results.

4.1 Schur–Weyl duality

Schur–Weyl duality is a result that gives a precise relationship between the representation theory of the symmetric group and the representation theory of the general linear group.

Fix $n \geq 1$. The *general linear group* $\mathrm{GL}_n(\mathbb{C})$ is the group of invertible $n \times n$ complex matrices, under multiplication. It is an important example of a *Lie group*. The group $\mathrm{GL}_n(\mathbb{C})$ acts naturally on the space $\mathbb{C}^n$ (thought of as consisting of column vectors) via matrix multiplication. It then acts on the space

$$V := \underbrace{\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n}_{k \text{ factors}} \tag{4.1}$$

by simultaneous matrix multiplication:

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots gv_k, \quad v_1, \dots, v_k \in \mathbb{C}^n,\ g \in G,$$

extended by linearity.

On the other hand, the symmetric group $\mathfrak{S}_k$ also acts naturally on V by permuting the factors

$$\pi(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_{\pi^{-1}(1)} \otimes v_{\pi^{-1}(2)} \otimes \cdots \otimes v_{\pi^{-1}(k)}, \quad v_1, \dots, v_k \in \mathbb{C}^n,\ \pi \in \mathfrak{S}_k,$$

extended by linearity.

It is straightforward to verify that the actions of $\mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{S}_k$ on V commute:

$$g\pi v = \pi g v, \quad \text{for all } g \in \mathrm{GL}_n(\mathbb{C}),\ \pi \in \mathfrak{S}_k,\ v \in V.$$

Thus we have maps to the commutants:

$$\mathrm{GL}_n(\mathbb{C}) \to \mathrm{End}_{\mathfrak{S}_k} V \qquad \text{and} \qquad \mathfrak{S}_k \to \mathrm{End}_{\mathrm{GL}_n(\mathbb{C})} V.$$

Schur–Weyl duality asserts that the images of each of these maps generate the codomain (as an algebra). Furthermore, V decomposes as

$$V = \bigoplus_{\lambda} S^\lambda \otimes L^\lambda, \tag{4.2}$$

where the sum is over all Young diagrams (equivalently, partitions) with k boxes and at most n rows. In this decomposition, $\mathfrak{S}_k$ acts on the first factor, while $\mathrm{GL}_n(\mathbb{C})$ acts on the second factor. The L^λ are pairwise inequivalent irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ (just as the S^λ are pairwise inequivalent irreducible representations of $\mathfrak{S}_k$). It follows, for instance, that the multiplicity of S^λ in V is $\dim L^\lambda$, and that the multiplicity of L^λ in V is $\dim S^\lambda$.

Example 4.1.1. Suppose that $k = 2$ and $n \geq 2$. We know that $\mathfrak{S}_2$ has exactly two irreducible representations: the trivial representation and the sign representation. Thus we have

$$\mathbb{C}^n \otimes \mathbb{C}^n = S^2\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n,$$

where $S^2\mathbb{C}^n$ is the space of *symmetric tensors* (the subspace of $\mathbb{C}^n \otimes \mathbb{C}^n$ on which $\mathfrak{S}_2$ acts trivially) and $\Lambda^2\mathbb{C}^n$ is the space of *antisymmetric tensors* (the subspace of $\mathbb{C}^n \otimes \mathbb{C}^n$ on which $\mathfrak{S}_2$ acts via the sign representation). Each of these summands is an irreducible representation of $\mathrm{GL}_n(\mathbb{C})$.

4.2 Categorification

Categorification is a powerful tool for relating mathematical structures that may appear on the surface to be completely unrelated. It also reveals hidden mathematical structure and provides a method to study and organize the representation theory of important algebras. In this section we will give a very brief overview of some examples of categorification, including categorification of symmetric functions, bosonic Fock space, and representations of certain Lie algebras. For further details we refer the reader to the expository references [Kle05, LS12, Sav17] and to the original research papers [Gei77, LS13, Kho14, RS17]

4.2.1 Symmetric functions

For $n \geq 0$, let Sym_n denote the space of all degree n elements of

$$\mathbb{C}[\![x_1, x_2, \dots]\!]$$

that remain invariant under any permutation of the indeterminates $x_1, x_2, \dots$. For example, we have the *n-th power sum*

$$p_n = \sum_{i=1}^{\infty} x_i^n \in \mathrm{Sym}_n, \quad n \in \mathbb{Z}_{>0}.$$

We set $p_0 = 1$. The algebra of *symmetric functions* is

$$\mathrm{Sym} := \bigoplus_{n=0}^{\infty} \mathrm{Sym}_n,$$

together with the natural sum and product of formal power series.

Let $\mathcal{P}$ denote the set of all partitions, including the empty partition $\varnothing$ of 0. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathcal{P}$, we define the *power sum symmetric function*

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}.$$

One can show that $\{p_\lambda : \lambda \in \mathcal{P}\}$ is a basis for Sym, so that

$$\mathrm{Sym} = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{C} p_\lambda,$$

It follows that Sym is isomorphic as an algebra to the polynomial algebra in the p_n:

$$\mathrm{Sym} \cong \mathbb{C}[p_1, p_2, \dots].$$

We define an inner product on Sym by declaring

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda, \qquad z_\lambda = \prod_{i=1}^{\lambda_1} i^{m_i(\lambda)} m_i(\lambda)!,$$

where $m_i(\lambda)$ denotes the number of parts of the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ equal to i. (Note that $|\lambda|!/z_\lambda$ is the number of partitions of $|\lambda|$ that have cycle type λ. See Exercise 3.1.1.)

For $\lambda \in \mathcal{P}$, a *semistandard tableau* of shape of λ is a filling T of the boxes of λ (considered as a Young diagram) with elements of $\mathbb{Z}_{>0}$ such that the entries are weakly increasing from left to right along rows and strictly increasing down columns. For example

1	1	2	2	3	5
2	3	3	3	6	
3	4				
6					

is a semistandard tableau of shape $(6, 5, 2, 1)$.

For a semistandard tableau T, define

$$x^T := \prod_{i=1}^{\infty} x_i^{t_i} \in \mathbb{C}[\![x_1, x_2, \dots]\!],$$

where t_i is the number of occurrences of i in the tableau T. (Note that the above product is actually finite since $t_i = 0$ for all but finitely many values of i.)

The *Schur function* corresponding to $\lambda \in \mathcal{P}$ is

$$s_\lambda := \sum_T x^T,$$

where the sum is over all semistandard tableaux of shape λ. It can be shown that the Schur functions form an orthonormal basis for Sym.

For $n \in \mathbb{Z}_{>0}$, let

$$h_n := s_{(n)} = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

We define $h_0 = 1$. For every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ we have the corresponding *complete symmetric function*

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}.$$

One can show that $\{h_\lambda : \lambda \in \mathcal{P}\}$ is a basis for Sym. It follows that

$$\mathrm{Sym} = \mathbb{C}[h_1, h_2, \ldots].$$

(In fact, the complete symmetric functions have the slightly better property that they generate Sym over $\mathbb{Z}$, which makes them more naturally suited to categorification.)

4.2.2 The Grothendieck group

For the remainder of these notes, we let $\mathfrak{S}_0$ denote the trivial group, so that $\mathfrak{S}_0 = \mathfrak{S}_1$. Clearly, $\mathfrak{S}_0$ has one irreducible representation, which we denote by $S^\varnothing$, where $\varnothing$ is the empty partition of 0.

The (finite-dimensional) representations of $\mathfrak{S}_n$, together with homomorphisms of representations, form a *category* $\mathrm{Rep}\,\mathfrak{S}_n$. For a representation V of $\mathfrak{S}_n$, let $[V]$ denote its isomorphism class. Consider the free vector space F_n on the set of isomorphism classes of representations of $\mathfrak{S}_n$:

$$F_n := \bigoplus_{[V]} \mathbb{C}[V], \tag{4.3}$$

where the sum is over all isomorphism classes $[V]$.

Now define

$$\tilde{F}_n = \langle [V_1 \oplus V_2] - [V_1] - [V_2] : V_1, V_2 \text{ reps of } \mathfrak{S}_n \rangle \subseteq F_n. \tag{4.4}$$

The *Grothendieck group* of $\mathrm{Rep}\,\mathfrak{S}_n$ is defined to be

$$K(\mathfrak{S}_n) := F_n/\tilde{F}_n.$$

Equivalently, $K(\mathfrak{S}_n)$ is the quotient of F_n by the relation

$$[V_1 \oplus V_2] = [V_1] + [V_2], \quad \text{for all representations } V_1 \text{ and } V_2 \text{ of } \mathfrak{S}_n.$$

We will denote the image in $K(\mathfrak{S}_n)$ of an isomorphism class $[V]$ again by $[V]$. The group operation on $K(\mathfrak{S}_n)$ is the vector space addition.

One can show that $K(\mathfrak{S}_n)$ has a basis given by the classes of the irreducible representations. Thus

$$K(\mathfrak{S}_n) = \bigoplus_{\lambda \vdash n} \mathbb{C}[S^\lambda].$$

Define

$$K(\mathfrak{S}) := \bigoplus_{n=0}^{\infty} K(\mathfrak{S}_n),$$

so that

$$K(\mathfrak{S}) = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{C}[S^\lambda].$$

Taking the Grothendieck group of the category Rep $\mathfrak{S}_n$ is an example of *decategorification*. It takes a category and produces a vector space. One can take the Grothendieck group of other categories, provided they have enough structure, but that is beyond the scope of our current discussion.

4.2.3 Categorification of the algebra of symmetric functions

Recall that for $\ell, k \in \mathbb{Z}_{\geq 0}$, we can view $\mathfrak{S}_\ell \times \mathfrak{S}_k$ as a subgroup of $\mathfrak{S}_{\ell+k}$, where $\mathfrak{S}_\ell$ permutes the elements $\{1, 2, \dots, \ell\}$ and $\mathfrak{S}_k$ permutes the elements $\{\ell+1, \ell+2, \dots, \ell+k\}$. Therefore, given a representation U of $\mathfrak{S}_\ell$ and a representation V of $\mathfrak{S}_k$, we have the outer tensor product representation $U \boxtimes V$ of $\mathfrak{S}_\ell \times \mathfrak{S}_k$ (see Section 1.1.7), and hence the induced representation

$$\mathrm{Ind}_{\mathfrak{S}_\ell \times \mathfrak{S}_k}^{\mathfrak{S}_{\ell+k}}(U \boxtimes V)$$

of $\mathfrak{S}_{\ell+k}$. Since the operations of outer tensor product and induction preserve isomorphism and direct sums, we have an induced bilinear map

$$\left[\mathrm{Ind}_{\mathfrak{S}_\ell \times \mathfrak{S}_k}^{\mathfrak{S}_{\ell+k}}\right] : K(\mathfrak{S}_\ell) \otimes K(\mathfrak{S}_k) \to K(\mathfrak{S}_{\ell+k}).$$

Taking these maps for all ℓ, k gives a binary operation on $K(\mathfrak{S})$. One can show that this operation is associative and unital. The unit element is class of the trivial representation of $K(\mathfrak{S}_0)$.

The following theorem was first proved by Geissinger in [Gei77].

Theorem 4.2.1 (Categorification of the algebra of symmetric functions)**.** *The linear map*

$$\Phi \colon K(\mathfrak{S}) \to \mathrm{Sym}$$

determined by $\Phi([S^\lambda]) = s_\lambda$, $\lambda \in \mathcal{P}$, *is an isomorphism of algebras.*

Theorem 4.2.1 is an example of a *categorification*. We have a category (namely, the category of representations of symmetric groups) with some extra structure coming from induction. Decategorifying, i.e. passing to the Grothendieck group, recovers the algebra of symmetric functions. Thus, the category of representations of symmetric groups *categorifies* the algebra of symmetric functions. In fact, if we also consider the restriction functors $\mathrm{Res}_{\mathfrak{S}_\ell \times \mathfrak{S}_k}^{\mathfrak{S}_{\ell+k}}$, we obtain the structure of a *coproduct* on $K(\mathrm{Sym})$. Then Theorem 4.2.1 can be strengthened to state that we have an isomorphism of *Hopf algebras*.

We can also categorify the inner product on Sym as follows. For U, V representations of $\mathfrak{S}_n$, we define

$$\langle [U], [V] \rangle = \dim \mathrm{Hom}_{\mathfrak{S}_n}(U, V). \tag{4.5}$$

One can show that this does indeed define an inner product on $K(\mathfrak{S})$ (we declare elements of $K(\mathfrak{S}_n)$ to be orthogonal to elements of $K(\mathfrak{S}_m)$ for $n \neq m$). For $\lambda, \mu \in \mathcal{P}$, it follows from Schur's lemma that

$$\langle [S^\lambda], [S^\mu] \rangle = \delta_{\lambda,\mu}.$$

Thus, the $[S^\lambda]$, $\lambda \in \mathcal{P}$, form an orthonormal basis for $K(\mathfrak{S})$. It follows that the isomorphism Φ of Theorem 4.2.1 is an isometry (i.e. it respects the inner products).

4.2.4 The Heisenberg algebra

Heisenberg algebras play a fundamental role in mathematics and mathematical physics. The (*infinite rank*) *Heisenberg algebra* H is the unital associative $\mathbb{C}$-algebra with generators p_n, p_n^*, $n \in \mathbb{Z}_{>0}$, and relations

$$p_n p_m^* = p_m^* p_n + \delta_{n,m} 1, \quad p_n p_m = p_m p_n, \quad p_n^* p_m^* = p_m^* p_n^*, \quad n, m \in \mathbb{Z}_{>0}. \tag{4.6}$$

The first relation in (4.6) is often called the *canonical commutation relation* in the physics literature, where the generators p_n^* and p_n correspond to position and momentum operators in a single particle system with a countable infinite number of degrees of freedom. The Heisenberg algebra is also crucial in the study of the quantum harmonic oscillator.

There is another, more presentation independent, way to describe the Heisenberg algebra H. Any $f \in \mathrm{Sym}$ acts on Sym via multiplication. Let f^* denote the operator on Sym adjoint to multiplication by f:

$$\langle f^*(g), h\rangle = \langle g, fh\rangle \quad \text{for all } f, g, h \in \mathrm{Sym}.$$

Then H is the subalgebra of $\mathrm{End}_{\mathbb{C}} \mathrm{Sym}$ generated by the operators f and f^*, for $f \in \mathrm{Sym}$. The tautological action of H on Sym is called the *bosonic Fock space representation*. Any choice of generating set for Sym yields a presentation of H. In particular, if we choose power sums, we recover the presentation (4.6). If we instead choose the complete symmetric functions, we see that H is the unital associative $\mathbb{C}$-algebra generated by h_n, h_n^*, $n \in \mathbb{Z}_{>0}$, are relations

$$h_n h_m^* = \sum_{r=0}^{\min(m,n)} h_{m-r}^* h_{n-r}, \quad h_n h_m = h_m h_n, \quad h_n^* h_m^* = h_m^* h_n^* \quad n, m \in \mathbb{Z}_{>0}. \tag{4.7}$$

Note, in particular, that h_1 and h_1^* satisfy the canonical commutation relation:

$$h_1 h_1^* = h_1^* h_1 + 1 \tag{4.8}$$

4.2.5 Categorification of bosonic Fock space

By Corollary 3.3.10, for $\lambda \vdash n$, we have

$$\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} S^\lambda = \bigoplus_{\substack{\mu \vdash n-1: \\ \lambda \to \mu}} S^\mu \qquad \text{and} \qquad \mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} S^\lambda = \bigoplus_{\substack{\mu \vdash n+1: \\ \mu \to \lambda}} S^\mu.$$

Restriction and induction respect isomorphism, in the sense that, if $V_1 \cong V_2$ are isomorphic representations of $\mathfrak{S}_n$, then

$$\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} V_1 \cong \mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} V_2 \qquad \text{and} \qquad \mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} V_1 \cong \mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} V_2.$$

Thus, restriction and induction induce linear maps (see (4.3)):

$$[\mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}] \colon F_n \to F_{n+1}, \qquad [\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}] \colon F_n \to F_{n-1}.$$

Restriction and induction also respect direct sums, in the sense that

$$\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(V_1 \oplus V_2) \cong \left(\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} V_1\right) \oplus \left(\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} V_2\right), \text{ and}$$
$$\operatorname{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V_1 \oplus V_2) \cong \left(\operatorname{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n} V_1\right) \oplus \left(\operatorname{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n} V_2\right).$$

It follows that (see (4.4))

$$[\operatorname{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}](\tilde{F}_n) \subseteq \tilde{F}_{n+1}, \qquad [\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}](\tilde{F}_n) \subseteq \tilde{F}_{n-1}.$$

Therefore, we have induced maps on the quotients:

$$[\operatorname{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}] \colon K(\mathfrak{S}_n) \to K(\mathfrak{S}_{n+1}), \qquad [\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}] \colon K(\mathfrak{S}_n) \to K(\mathfrak{S}_{n-1}).$$

Then we have linear maps

$$[\operatorname{Ind}] \colon K(\mathfrak{S}) \to K(\mathfrak{S}), \quad [\operatorname{Ind}] = \sum_{n=0}^{\infty} \left[\operatorname{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}\right]$$

and

$$[\operatorname{Res}] \colon K(\mathfrak{S}) \to K(\mathfrak{S}), \quad [\operatorname{Res}] = \sum_{n=1}^{\infty} \left[\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\right]$$

where we adopt the convention that [Res] acts as zero on $K(\mathfrak{S}_0)$.

Using the combinatorics of the Young graph, one can show that, for all $\lambda \vdash n$, $n \geq 1$, we have

$$\operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\lambda \cong \left(\operatorname{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n} \operatorname{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n} S^\lambda\right) \oplus S^\lambda.$$

(See Exercise 4.2.1.) It follows that

$$[\operatorname{Ind}][\operatorname{Res}] = [\operatorname{Res}][\operatorname{Ind}] + 1 \tag{4.9}$$

as linear operators on $K(\mathfrak{S})$. Note that these are precisely the canonical commutation relations satisfied by the generators e_1 and e_1^* of the Heisenberg algebra (see (4.8))!

What about other generators? Recall from Section 4.2.3 that we have defined a product on $K(\mathfrak{S})$. Thus, for $n \in \mathbb{Z}_{>0}$ we have the operator given by multiplication by $[S^{(n)}]$:

$$a_n : K(\mathfrak{S}) \to K(\mathfrak{S}), \quad a_n([V]) = [V] \cdot [S^{(n)}] = \left[\operatorname{Ind}^{\mathfrak{S}_{k+n}}_{\mathfrak{S}_k \times \mathfrak{S}_n} \left(V \boxtimes S^{(n)}\right)\right], \quad V \text{ a rep of } \mathfrak{S}_k.$$

Since we also have an inner product on $K(\mathfrak{S})$ as in (4.5), we can consider operators a_n^* adjoint to a_n. These can also be described directly using restriction. One can then show that these operators satisfy the relations (4.7):

$$a_n a_m^* = \sum_{r=0}^{\min(m,n)} a_{m-r}^* a_{n-r}, \quad a_n a_m = a_m a_n, \quad a_n^* a_m^* = a_m^* a_n^*, \quad n, m \in \mathbb{Z}_{>0}. \tag{4.10}$$

It follows that we have an action of the Heisenberg algebra on $K(\mathrm{Sym})$. In other words, we can view $K(\mathrm{Sym})$ as a representation of H

Theorem 4.2.2 (Categorification of bosonic Fock space)**.** *The map Φ of Theorem 4.2.1 is an isomorphism of representations of the Heisenberg algebra H from $K(\mathfrak{S})$ to the bosonic Fock space representation of H on* Sym*.*

Exercises.

4.2.1. Prove that for all $\lambda \vdash n$, $n \geq 1$, we have

$$\operatorname{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} \operatorname{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} S^\lambda \cong \left(\operatorname{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \operatorname{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} S^\lambda\right) \oplus S^\lambda.$$

4.2.6 Categorification of the basic representation

The categorification of bosonic Fock space described in Section 4.2.5 can be refined somewhat. Suppose V is a representation of $\mathfrak{S}_n$ for some $n \in \mathbb{Z}_{>0}$. Since the YJM element X_n commutes with $\mathfrak{S}_{n-1}$ (see Exercise 3.2.1), the action of X_n on $\operatorname{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} V$ gives an $\mathfrak{S}_{n-1}$-intertwiner. We know that X_n acts diagonally, with integral eigenvalues. For $i \in \mathbb{Z}$, let proj_i denote the projection onto the eigenspace corresponding to eigenvalue i. If we define

$$\operatorname{Res}_i V := \operatorname{proj}_i \operatorname{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} V,$$

it follows that

$$\operatorname{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} V = \bigoplus_{i \in \mathbb{Z}} \operatorname{Res}_i V.$$

(Note that $\operatorname{Res}_i V = 0$ for all but finitely many values of i.) We call Res_i *i-restriction*. This is a functor, with an adjoint functor *i-induction*, denoted Ind_i. In terms of the combinatorics of standard tableaux, Res_i acts on irreducible representations by removing a box of content i (if such a box exists). That is,

$$\operatorname{Res}_i S^\lambda = \begin{cases} S^\mu & \text{if } \mu \text{ is obtained from } \lambda \text{ by removing a box of content } i, \\ 0 & \text{if } \mu \text{ has no removable boxes of content } i. \end{cases}$$

Similarly, Ind_i adds a box of content i (if possible).

As in Section 4.2.5, we have induced maps

$$[\operatorname{Ind}_i], [\operatorname{Res}_i] \colon K(\mathfrak{S}) \to K(\mathfrak{S}).$$

Let $\mathfrak{sl}_\infty$ denote the space of trace zero $\mathbb{Z} \times \mathbb{Z}$ matrices with a finite number of nonzero entries. In other words, elements of $\mathfrak{sl}_\infty$ are matrices $X = (X_{i,j})_{i,j \in \mathbb{Z}}$ such that $X_{i,j} = 0$ for all but finitely many pairs $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, and such that $\sum_{i \in \mathbb{Z}} X_{i,i} = 0$. This is a *Lie algebra* with Lie bracket

$$[X, Y] = XY - YX,$$

where juxtaposition denotes matrix multiplication.

One can prove that the action of $[\operatorname{Ind}_i]$ and $[\operatorname{Res}_i]$ on $K(\mathfrak{S})$ define an action of the Lie algebra $\mathfrak{sl}_\infty$ on $K(\mathfrak{S})$. The particular representation one obtains is called the *basic representation*. Thus, the representation theory of symmetric groups yields a categorification of the basic representation of $\mathfrak{sl}_\infty$.

4.2.7 Going even further

The constructions discussed briefly above are just the start of the deep and extremely active area of categorification. We mention here some related constructions. For details on the first two, we refer the reader to the book [Kle05].

Positive characteristic. If we work over a field of positive characteristic p instead of over the complex numbers, we can repeat the categorification of the basic representation sketched in Section 4.2.6. Then the eigenvalues of the YJM elements lie in $\mathbb{Z}/p\mathbb{Z}$. We obtain a categorification of the affine Lie algebra $\widehat{\mathfrak{sl}}_p$. Note that, in positive characteristic, Maschke's Theorem fails (see Exercise 1.1.12). It is no longer true that every representation of $\mathfrak{S}_n$ decomposes as a sum of irreducible ones. So here the representation theory is much more complicated. But categorification provides some very useful tools for studying these representations.

Higher level cyclotomic quotients. The group algebra $L(\mathfrak{S}_n)$ of the symmetric group is a quotient of the *degenerate affine Hecke algebra* (see Exercise 3.2.3) by the ideal generated by x_1. More generally one can take the *cyclotomic quotient* by the ideal generated by

$$\prod_{i\in\mathbb{Z}}(x_1 - i)^{\mu_i},$$

for some $\mu_i \in \mathbb{Z}_{\geq 0}$, $i \in \mathbb{Z}$, where all but finitely many of the μ_i are equal to zero. Such a quotient is called a *degenerate cyclotomic Hecke algebra*. One can repeat the constructions of Sections 4.2.5 and 4.2.6 in this setting and obtain categorifications of other representations of the Heisenberg algebra and of $\mathfrak{sl}_\infty$ (or of $\widehat{\mathfrak{sl}}_p$ if we work in positive characteristic). The study of these categorification is currently an active area research.

Wreath product algebras. One can replace the group algebra $L(\mathfrak{S}_n)$ by *wreath product algebras*. These are algebras of the form

$$F^{\otimes n} \otimes L(\mathfrak{S}_n),$$

where the multiplication involves the action of $\mathfrak{S}_n$ on $F^{\otimes n}$ by permuting the factors (similar to a semidirect product of groups). We have a chain

$$\mathbb{C} \subseteq F \subseteq F^{\otimes 2} \otimes L(\mathfrak{S}_2) \subseteq F^{\otimes 3} \otimes L(\mathfrak{S}_3) \subseteq \cdots.$$

Then one can examine the functors of induction and restriction. There are deep connections to representation, geometry, and algebraic combinatorics. We refer the reader to [RS17] for further details.

Index

Bibliography

[CSST10] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli. *Representation theory of the symmetric groups*, volume 121 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. The Okounkov-Vershik approach, character formulas, and partition algebras. doi:10.1017/CBO9781139192361.

[Gei77] L. Geissinger. Hopf algebras of symmetric functions and class functions. pages 168–181. Lecture Notes in Math., Vol. 579, 1977.

[Jud] T. W. Judson. *Abstract algebra: theory and applications*. URL: http://abstract.ups.edu/.

[Kho14] M. Khovanov. Heisenberg algebra and a graphical calculus. *Fund. Math.*, 225(1):169–210, 2014. Available at https://arxiv.org/abs/1009.3295. doi:https://doi.org/10.4064/fm225-1-8.

[Kle05] A. Kleshchev. *Linear and projective representations of symmetric groups*, volume 163 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2005. URL: https://doi.org/10.1017/CBO9780511542800.

[LS12] A. Licata and A. Savage. A survey of Heisenberg categorification via graphical calculus. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 7(2):291–321, 2012.

[LS13] A. Licata and A. Savage. Hecke algebras, finite general linear groups, and Heisenberg categorification. *Quantum Topol.*, 4(2):125–185, 2013. URL: https://doi.org/10.4171/QT/37.

[Mur83] G. E. Murphy. The idempotents of the symmetric group and Nakayama's conjecture. *J. Algebra*, 81(1):258–265, 1983. doi:10.1016/0021-8693(83)90219-3.

[OV96] A. Okounkov and A. Vershik. A new approach to representation theory of symmetric groups. *Selecta Math. (N.S.)*, 2(4):581–605, 1996. doi:10.1007/PL00001384.

[Py03] P. Py. On representation theory of symmetric groups. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 301(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 9):229–242, 245–246, 2003. doi:10.1007/s10958-005-0316-7.

[Roy] D. Roy. Linear algebra II: Notes for MAT 3141. Translated by A. Savage. URL: http://alistairsavage.ca/mat3141/notes/MAT%203141%20-%20Linear%20Algebra%20II.pdf.

[RS17] D. Rosso and A. Savage. A general approach to Heisenberg categorification via wreath product algebras. *Math. Z.*, 286(1-2):603–655, 2017. URL: https://doi.org/10.1007/s00209-016-1776-9.

[Sav17] A. Savage. Heisenberg categorification. *CMS Notes*, 49(3):16–17, 2017. Available at https://cms.math.ca/notes/vault/Notesv49n3.pdf.

[Tri] S. Triel. *Linear algebra done wrong*. URL: http://www.math.brown.edu/~treil/papers/LADW/LADW.html.

[Ver06] A. M. Vershik. A new approach to the representation theory of the symmetric groups. III. Induced representations and the Frobenius-Young correspondence. *Mosc. Math. J.*, 6(3):567–585, 588, 2006.

[VO04] A. M. Vershik and A. Yu. Okounkov. A new approach to representation theory of symmetric groups. II. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 307(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 10):57–98, 281, 2004. doi:10.1007/s10958-005-0421-7.