University of Ottawa

Department of Mathematics and Statistics

MAT 4141/5141: (Topics in) Algebra I
Professor: Alistair Savage

Midterm Test
November 7, 2018

Surname _____ First Name _____

Student # _____

Instructions:

(a) You have 80 minutes to complete this exam.
(b) Unless otherwise indicated, you must justify your answers to receive full marks.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	5	3	4	5	4	3	24
Grade							

QUESTION 1. [**5 points**]

(a) State the Hilbert Basis Theorem.

(b) Given an example of a ring R such that

- R is noetherian,
- R is not a principal ideal domain, and
- R is not artinian.

Justify your answer. (To show that R is not a principal ideal domain, you may simply give a non-principal ideal. You do not need to justify in detail that this ideal is not principal.)

(c) Give an example of a ring that is neither noetherian nor artinian. Justify your answer.

QUESTION 2. [**3 points**] Suppose R is a commutative ring and S is a multiplicative subset of R. Prove that the natural ring homomorphism $h\colon R \to S^{-1}R$ is injective if and only if S contains no zero divisors.

QUESTION 3. **[4 points]** Fix a field k. Suppose $X \subseteq \mathbb{A}^n$ and $S \subseteq k[X_1, \ldots, X_n]$.

(a) Define the variety $V(S)$ and the vanishing ideal $I(X)$.

(b) Prove that $IV(S) \supseteq S$.

(c) Prove that $VIV(S) = V(S)$. (You may use without proof the fact that the maps $S \mapsto V(S)$ and $X \mapsto I(X)$ are inclusion-reversing.)

(d) Now suppose $n = 1$ and $k = \mathbb{Q}$, and consider the ideal

$$I = \langle X^3(X-1)^2(X+2)\rangle \subseteq \mathbb{Q}[X].$$

What is $\sqrt{I}$? You do not need to justify your answer.

(e) **Bonus (+2 points)**: Why is the assumption $k = \mathbb{Q}$ important above? Give a field k over which your answer to the previous part would be different.

QUESTION 4. **[5 points]** Suppose that I and J are ideals of a commutative ring R.

(a) Show that every element of $R/I \otimes_R R/J$ can be written in the form $(r + I) \otimes (1 + J)$ for some $r \in R$.

(b) Show that

$$(R/I) \otimes_R (R/J) \cong R/(I + J)$$

as R-modules.

QUESTION 5. [**4 points**] Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the standard basis elements of $\mathbb{R}^2$. Show that

$$e_1 \otimes e_1 + e_2 \otimes e_2 \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$$

is not a simple tensor. That is, show that it cannot be written in the form $v \otimes w$ for some $v, w \in \mathbb{R}^2$.

QUESTION 6. [**3 points**] Suppose that k is an algebraically closed field and R is a k-algebra. Furthermore, suppose M, N_1, N_2 are simple R-modules that are all pairwise isomorphic and finite dimensional over k. Show that

$$\mathrm{Hom}_R(M, N_1 \oplus N_2) \cong k^2 \quad \text{as } k\text{-vector spaces.}$$

Here the k-vector space structure on $\mathrm{Hom}_R(M, N_1 \oplus N_2)$ is given by

$$(af)(x) = af(x), \quad a \in k,\ f \in \mathrm{Hom}_R(M, N_1 \oplus N_2),\ x \in M.$$

(You do not need to verify that this defines the structure of a vector space.)

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