uOttawa
Department of Mathematics
and Statistics
Département de mathématiques
et statistique

Analysis III – Mat 3120

Mid-Term Exam — June 18, 2014

Time: 1 hour 20 minutes.
Attempt all questions in (1)–(3).
The problem ($4\star$) is optional.
Total number of marks: 18 (+ 2 bonus marks).

(1) (a) [**1 point**] Give the definition of a metric on a set X.
(b) [**2 points**] Let X be a set containing at least two distinct elements. Prove that there are infinitely many metrics on X that are pairwise different.
(c) [**1 point**] State what it means that a metric space X isometrically embeds into a metric space Y.
(d) [**2 points**] Let X be a finite set with n points equipped with the 0-1-distance. Prove that the metric space X isometrically embeds into the n-dimensional Euclidean space $\ell^2(n)$.

(2) (a) [**1 point**] State the definition of an open subset of a metric space.
(b) [**2 points**] Prove that the set

$$P = \{x \in \ell^\infty \colon \forall i = 1, 2, \ldots,\ x_i > 0\}$$

is *not* open in the metric space ℓ^∞.
(c) [**1 point**] State the definition of the interior of a set in a metric space.
(d) [**2 points**] Describe, with a proof, the interior of the set from question (2b).

[Continued overleaf....]

(3) (a) [**1 point**] Give the definition of a connected metric space.

(b) [**2 points**] Let X be a metric space containing an everywhere dense connected subspace Y. Prove that X is connected.

(c) [**1 point**] Give the definition of a path-connected metric space.

(d) [**2 points**] Now let X be a metric space containing an everywhere dense path-connected subspace Y. Can one conclude that X is necessarily path-connected? Explain.

(4) [$\star$ **bonus question — 2 points**] Let X be a separable metric space. Prove that the collection of all open subsets of X has cardinality not exceeding $\mathfrak{c} = 2^{\aleph_0}$.
$\overset{\infty}{\smile}$
$*$

[End of the exam questions]