Analysis III – Mat 3120

Solutions to the 2014 mid-term exam

(1) (a) [**1 point**] Give the definition of a metric on a set X.

Reference. See Definition 1.1. □

(b) [**2 points**] Let X be a set containing at least two distinct elements. Prove that there are infinitely many metrics on X that are pairwise different.

Solution. For instance, one can take the family of all the multiples of the zero-one distance:

$$d_\lambda(x, y) = \lambda d_{0-1}(x, y).$$

Clearly, each d_λ is a metric whenever $\lambda \in \mathbb{R}$ and $\lambda > 0$.
Since by the assumption there are at least two distinct points, $x, y \in X$, $x \neq y$, one has for every $\lambda > 0$

$$d_\lambda(x, y) = \lambda,$$

so if $\lambda \neq \mu$,

$$d_\lambda(x, y) \neq d_\mu(x, y),$$

so one concludes that the metric d_λ is distinct from the metric d_μ. □

(c) [**1 point**] State what it means that a metric space X isometrically embeds into a metric space Y.

Reference. See Definition 2.1. □

(d) [**2 points**] Let X be a finite set with n points equipped with the 0-1-distance. Prove that the metric space X isometrically embeds into the n-dimensional Euclidean space $\ell^2(n)$.

Solution. Enumerate the points of X:

$$X = \{x_1, x_2, \dots, x_n\}$$

and define a mapping $i\colon X \to \mathbb{R}^n$ by

$$i(x_i) = \frac{\sqrt{2}}{2} e_i,$$

where e_i is the i-th standard basic vector in $\mathbb{R}^n$. To verify that i is an isometric embedding, let $x, y \in X$. Then for some i, j one has $x = x_i$, $y = y_j$. If $i = j$, there is nothing to verify, so suppose $i \neq j$. We have

$$\| i(x_i), i(x_j) \| = \frac{\sqrt{2}}{2} \| e_i - e_j \| = \frac{\sqrt{2}}{2} \cdot \sqrt{2} = 1 = d_{0-1}(x_i, x_j),$$

as required.
This problem appeared in the lecture notes as Exercise 2.7. □

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(2) (a) [**1 point**] State the definition of an open subset of a metric space.

Reference. See Definition 2.8. □

(b) [**2 points**] Prove that the set

$$P = \{x \in \ell^\infty \colon \forall i = 1, 2, \dots,\ x_i > 0\}$$

is *not* open in the metric space ℓ^∞.

Solution. Define an element $x \in \ell^\infty$ by the condition $x_n = 1/n$, $n \in \mathbb{N}_+$. Then no open ball around x is contained in A. Indeed, given $\epsilon > 0$, by Archimedes' property of $\mathbb{R}$ we can find k with $1/k < \epsilon$. The sequence y given by

$$y_n = \begin{cases} \frac{1}{n}, & \text{if } n \neq k, \\ 0, & \text{if } n = k, \end{cases}$$

belongs to the ball $B_\epsilon(x)$:

$$\| x - y \|_\infty = \frac{1}{k} < \epsilon,$$

yet $y \notin A$ since $y_k = 0$.
This was Exercise 2.20 in the lecture notes. Example 4.1 is very closely related to this problem. □

(c) [**1 point**] State the definition of the interior of a set in a metric space.

Solution. See Definitions 4.3 and 4.10. □

(d) [**2 points**] Describe, with a proof, the interior of the set from question (2b).

Solution. Analysing the counter-example in (2b), we come to the conclusion that the interior points of A are those positive sequences that are *uniformly bounded away from zero*:

$$x \in \operatorname{Int} A \iff \inf_n x_n > 0.$$

Let us verify this claim. First suppose that for an $x = (x_n)$, $\inf_n x_n > 0$. Denote $\epsilon = \inf_n x_n$. Then $B_\epsilon(x) \subseteq A$. Indeed, if $y \in B_\epsilon(x)$, then $\| x - y \|_\infty < \epsilon$, and for every n one has, by the triangle inequality,

$$y_n > x_n - \epsilon = 0,$$

meaning $y \in A$. This establishes the implication $\Leftarrow$.
The other implication is proved by contraposition. Suppose that $\inf_n x_n \leq 0$. If the inequality is strict, then for at least one n one must have $x_n < 0$, meaning $x \notin A$. Such an x cannot be an interior point of A either. So we can suppose $\inf_n x_n = 0$. Now proving that no open ball around x is contained in A is done pretty much along the same lines as in the above example (2b).
I will nevertheless repeat the argument. Let $\epsilon > 0$ be arbitrary. There is k so that $0 \leq x_k < \epsilon$. The sequence y defined by

$$y_n = \begin{cases} x_n, & \text{if } n \neq k, \\ 0, & \text{if } n = k \end{cases}$$

belongs to $B_\epsilon(x)$, and yet $y \notin A$. This argument shows that $x \notin \operatorname{Int} A$ either. □

(3) (a) [**1 point**] Give the definition of a connected metric space.

Reference. Definition 7.5 + Definition 7.8. □

(b) **[2 points]** Let X be a metric space containing an everywhere dense connected subspace Y. Prove that X is connected.

Solution. Let V be a non-empty proper open and closed subset of X. The subset $V' = V \cap Y$ of Y is open in Y, by one of our results. It is also closed in Y, by the same argument. Since Y is everywhere dense in X, the set V' is non-empty. Since $X \setminus V$ is non-empty and open in X, the set

$$(X \setminus V) \cap Y = Y \setminus V'$$

is non-empty, meaning that V' is proper. We conclude that Y is disconnected. By contraposition, we are done.
This was your Exercise 10.22 in the lecture notes. □

(c) **[1 point]** Give the definition of a path-connected metric space.

Reference. See Definition 9.5. □

(d) **[2 points]** Now let X be a metric space containing an everywhere dense path-connected subspace Y. Can one conclude that X is necessarily path-connected? Explain.

Solution. The answer is in the negative, and it is provided by the construction of a connected subset of the plane which is not path-connected (Example 10.23). In the notation of the example, X_2 is path-connected and everywhere dense in X, and yet X itself is not path-connected. It is enough just to give a reference to this example without reproducing it. □

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(4) **[★ bonus question — 2 points]** Let X be a separable metric space. Prove that the collection of all open subsets of X has cardinality not exceeding $\mathfrak{c} = 2^{\aleph_0}$.

Reference. Problem 4 from Assignment 5. Since almost everyone has produced a solution, I expect that almost everyone will give an impeccable solution of the bonus problem either. $\underset{*}{\overset{\infty}{\smile}}$ □