MAT 3341 – Spring/Summer 2019

Review

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IMPORTANT NOTE: This is only a partial review. The exam covers *all* topics in the course, not just topics appearing in this review.

FINAL EXAM

- 14 questions.
- 58 points
  - Definitions/comprehension/examples: ≈ 17.5 points
  - Computation: ≈ 27.5 points
  - Proofs: ≈ 13
- Covers entire course.
- You may write in pencil.

EXAM PERIOD OFFICE HOURS

- Tuesday, July 23, 11am–12pm
- Thursday, July 25, 11am–12pm

HERMITIAN AND UNITARY MATRICES

Definition (Hermitian matrix)**.** A square complex matrix A is *hermitian* if $A^H = A$, equivalently, if $\bar{A} = A^T$.

Properties of hermitian matrices:

- A matrix $A \in M_{n,n}(\mathbb{C})$ is hermitian if and only if
$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{C}^n.$$
- If $A \in M_{n,n}(\mathbb{C})$ is hermitian then every eigenvalue of A is real.
- If $A \in M_{n,n}(\mathbb{C})$ is hermitian, then eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

Recall that, in general, if $A \in M_{m,n}(\mathbb{C})$, then
$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^H\mathbf{y}\rangle \quad \text{for all } \mathbf{x} \in \mathbb{C}^n, \mathbf{y} \in \mathbb{C}^m.$$

Definition (Unitary matrix)**.** A matrix $U \in M_{n,n}(\mathbb{C})$ is *unitary* if is satisfies any one of the following equivalent conditions:

(a) U is invertible and $U^{-1} = U^H$.
(b) The rows of U are orthonormal.
(c) The columns of U are orthonormal.

Note that $U \in M_{n,n}(\mathbb{R})$ is unitary if and only if it is *orthogonal.*

Theorem (Schur's theorem). *If $A \in M_{n,n}(\mathbb{C})$, then there exists a unitary matrix U such that*

$$U^H AU = T$$

is upper triangular. Moreover, the entries on the main diagonal of T are the eigenvalues of A (including multiplicities).

Theorem (Spectral theorem). *Every hermitian matrix is unitarily diagonalizable. In other words, if A is a hermitian matrix, then there exists a unitary matrix U such that $U^H AU$ is diagonal.*

Theorem (Real spectral theorem, principal axes theorem). *The following conditions are equivalent for $A \in M_{n,n}(\mathbb{R})$.*

(a) A has an orthonormal set of n eigenvectors in $\mathbb{R}^n$.
(b) A is orthogonally diagonalizable. *That is, there exists a real orthogonal matrix P such that $P^{-1}AP = P^T AP$ is diagonal.*
(c) A is symmetric.

Definition (Normal matrix). An matrix $N \in M_{n,n}(\mathbb{C})$ is *normal* if $NN^H = N^H N$.

Every hermitian matrix is normal, but there are normal matrices that are not hermitian, e.g.

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

Theorem. *A complex square matrix is unitarily diagonalizable if and only if it is normal.*

Theorem (Cayley–Hamilton Theorem). *If $A \in M_{n,n}(\mathbb{C})$, then $c_A(A) = 0$. In other words, every square matrix is a "root" of its characteristic polynomial.*

Singular value decomposition

Definition (Singular value decomposition). A *singular value decomposition (SVD)* of $A \in M_{m,n}(\mathbb{C})$ is a factorization

$$A = P\Sigma Q^H,$$

where P and Q are unitary and, in block form,

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}_{m \times n}, \qquad D = \operatorname{diag}(d_1, d_2, \dots, d_r), \quad d_1, d_2, \dots, d_r \in \mathbb{R}_{>0}.$$

Algorithm (Finding a SVD). Suppose we want to find a SVD of $A \in M_{m,n}(\mathbb{C})$.

(a) Unitarily diagonalize the hermitian matrix $A^H A$. Find eigenvalues $\lambda_1, \dots, \lambda_n$ with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \quad \text{and} \quad \lambda_i = 0 \text{ for } i > r.$$

Find a corresponding orthonormal basis $\{\mathbf{q}_1, \dots, \mathbf{q}_n\}$ of eigenvectors. Then

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{d}_n \end{bmatrix} \in M_{n,n}(\mathbb{C})$$

is unitary and unitarily diagonalizes $A^H A$. Note that

- $r = \operatorname{rank} A$;

- the positive singular values of A^HA and AA^H are the same.

(b) The *singular values* of A are the real numbers

$$\sigma_i = \sqrt{\lambda_i} = \|A\mathbf{q}_i\|, \quad i = 1, 2, \ldots, n.$$

The matrix

$$\Sigma_A := \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}, \qquad \text{where} \quad D_A = \operatorname{diag}(\sigma_1, \ldots, \sigma_r),$$

is the *singular matrix* of A.

(c) Define

$$\mathbf{p}_i = \frac{1}{\sigma_i} A\mathbf{q}_i \quad \text{for } i = 1, 2, \ldots, r.$$

If $r < m$, extend $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_r\}$ to an orthonormal basis

$$\{\mathbf{p}_1, \ldots, \mathbf{p}_m\}$$

of $\mathbb{C}^m$. Define the unitary matrix

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_m \end{bmatrix} \in M_{m,m}(\mathbb{C}).$$

(d) A SVD for A is given by

$$A = P\Sigma_A Q^H.$$

Every matrix $A \in M_{m,n}(\mathbb{C})$ has a SVD. If $A \in M_{m,n}(\mathbb{R})$, then we can find a SVD where P and Q are *real* matrices.

What can we do with a SVD?

(a) We can easily see the rank of A: $\operatorname{rank} A = r$.

(b) We can easily describe the *fundamental subspaces* of A:
- $\{\mathbf{p}_1, \ldots, \mathbf{p}_r\}$ is an orthonormal basis of $\operatorname{col} A$,
- $\{\mathbf{p}_{r+1}, \ldots, \mathbf{p}_m\}$ is an orthonormal basis of $\operatorname{null} A^H$,
- $\{\mathbf{q}_{r+1}, \ldots, \mathbf{q}_n\}$ is an orthonormal basis of $\operatorname{null} A$,
- $\{\overline{\mathbf{q}_1}, \ldots, \overline{\mathbf{q}_r}\}$ is an orthonormal basis of $\operatorname{row} A$.

(c) We can easily compute pseudoinverses: $A^+ = Q\Sigma' P^H$, where

$$\Sigma' = \begin{bmatrix} D_A^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{n\times m}.$$

Jordan canonical form

If λ is an eigenvalue of A, then the associated eigenspace is

$$E_\lambda = \operatorname{null}(\lambda I - A) = \{\mathbf{x} : (\lambda I - A)\mathbf{x} = 0\} = \{\mathbf{x} : A\mathbf{x} = \lambda\mathbf{x}\}.$$

Definition (Generalized eigenspace)**.** If λ is an eigenvalue of the matrix A, then the associated *generalized eigenspace* is

$$G_\lambda = G_\lambda(A) := \operatorname{null}(\lambda I - A)^{m_\lambda},$$

where m_λ is the algebraic multiplicity of λ.

Note that we always have

$$E_\lambda(A) \subseteq G_\lambda(A).$$

Lemma. *If λ is an eigenvalue of A, then $\dim G_\lambda(A) = m_\lambda$, where m_λ is the algebraic multiplicity of λ.*

Recall the definition of *Jordan blocks*:

$$J(1,\lambda) = \begin{bmatrix}\lambda\end{bmatrix}, \; J(2,\lambda) = \begin{bmatrix}\lambda & 1 \\ 0 & \lambda\end{bmatrix}, \; J(3,\lambda) = \begin{bmatrix}\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda\end{bmatrix}, \; J(4,\lambda) = \begin{bmatrix}\lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda\end{bmatrix}, \text{ etc.}$$

Theorem (Jordan canonical form)**.** *Suppose $A \in M_{n,n}(\mathbb{C})$ has distinct (i.e. non-repeated) eigenvalues $\lambda_1, \dots, \lambda_k$. For $1 \le i \le k$, let m_i be the algebraic multiplicity of λ_i. Then there exists an invertible matrix P such that*

$$P^{-1}AP = \operatorname{diag}(J_1, \dots, J_m), \tag{1}$$

where each J_ℓ is a Jordan block corresponding to some eigenvalue λ_i. Furthermore, the sum of the sizes of the Jordan block corresponding to λ_i is equal to m_i. The form (1) is called a Jordan canonical form *of A.*

Example. If $A \in M_{4,4}(\mathbb{C})$ has only one eigenvalue λ (with multiplicity 4), its possible Jordan canonical forms (up to reordering the Jordan blocks) are

$$J(4,\lambda), \quad \operatorname{diag}(J(3,\lambda), J(1,\lambda)), \quad \operatorname{diag}(J(2,\lambda), J(2,\lambda)), \quad \operatorname{diag}(J(2,\lambda), J(1,\lambda), J(1,\lambda)),$$
$$\operatorname{diag}(J(1,\lambda), J(1,\lambda), J(1,\lambda), J(1,\lambda)) = \operatorname{diag}(\lambda, \lambda, \lambda, \lambda).$$

Only in the last case is A diagonalizable.