# MAT 3143 – Fall 2020

Midterm Exam

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*Your solutions should be submitted through Brightspace as a* ***single pdf file****. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

**This exam ends at 9:45am. You may not write anything on your pages after this time. You will then have until 9:55am to scan and submit your solutions on Brightspace.**

QUESTION 1 **(4 pts)**.

(a) State the definition of an *integral domain*.
(b) State the definition of a *division ring*.
(c) State the definition of an *idempotent element* of a ring.
(d) State the *Chinese Remainder Theorem*.

QUESTION 2 **(3 pts)**. Suppose $R$ is a ring containing a nilpotent unit $r$. Show that $R$ is the zero ring.

QUESTION 3 **(4 pts)**.

(a) State the definition of a *prime ideal* of a commutative ring.
(b) Suppose $R$ and $S$ are commutative rings, $f \colon R \to S$ is a ring homomorphism, and $P$ is a prime ideal of $S$. Show that
$$f^{-1}(P) := \{r \in R : f(r) \in P\}$$
is a prime ideal of $R$.

QUESTION 4 **(3 pts)**. Let $F$ be a field. Show that
$$I = \{a_0 + a_1x + \cdots + a_nx^n : n \geq 0,\ a_0, \dots, a_n \in F,\ a_0 + \cdots + a_n = 0\}$$
is an ideal of $F[x]$ and that $F[x]/I \cong F$ as rings.

QUESTION 5 **(3 pts)**. Prove that $f(x) = x^4 + 4x^3 + 3x^2 + 7x + 10$ is irreducible in $\mathbb{Q}[x]$. *Hint*: Consider $f(x-1)$.

QUESTION 6 **(4 pts)**. In each case, give the number of elements of $F[x]/\langle f(x)\rangle$, and state whether or not $F[x]/\langle f(x)\rangle$ is a field.

(a) $f(x) = x^3 - x + 1$, $F = \mathbb{Z}_3$.
(b) $f(x) = x^4 + x^3 + x + 1$, $F = \mathbb{Z}_2$.