MAT 3143 – Fall 2020

Midterm Exam – Solutions

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Your solutions should be submitted through Brightspace as a ***single pdf file****. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam ends at 9:45am. You may not write anything on your pages after this time. You will then have until 9:55am to scan and submit your solutions on Brightspace.

QUESTION 1 **(4 pts)**.

(a) State the definition of an *integral domain*.

Solution: An integral domain is a nonzero commutative ring with no zero divisors.

(b) State the definition of a *division ring*.

Solution: A division ring is a nonzero ring in which every nonzero element is a unit.

(c) State the definition of an *idempotent element* of a ring.

Solution: An element r in a ring is idempotent if $r^2 = r$.

(d) State the *Chinese Remainder Theorem*.

Solution: Let R be a ring and let A, B be two ideals of R such that $A + B = R$. Then $R/(A \cap B) \cong (R/A) \times (R/B)$.

QUESTION 2 **(3 pts)**. Suppose R is a ring containing a nilpotent unit r. Show that R is the zero ring.

Solution: Since r is nilpotent, there exists $n \in \mathbb{N}$ such that $r^n = 1$. Because r is a unit, there exists $s \in R$ such that $rs = sr = 1$. Then we have

$$1 = rs = r1s = rrss = \cdots = r^n s^n = 0s^n = 0.$$

Then, for all $a \in R$, we have

$$a = a1 = a0 = 0.$$

Hence R is the zero ring.

QUESTION 3 **(4 pts)**.

(a) State the definition of a *prime ideal* of a commutative ring.

Solution: An ideal P of a commutative ring R is prime if $P \neq R$ and

$$a, b \in R,\ ab \in P \implies a \in P \text{ or } b \in P.$$

(b) Suppose R and S are commutative rings, $f\colon R \to S$ is a ring homomorphism, and P is a prime ideal of S. Show that

$$f^{-1}(P) := \{r \in R : f(r) \in P\}$$

is a prime ideal of R.

Solution: We first show that $f^{-1}(P) \neq R$. Suppose, towards a contradiction that $f^{-1}(P) = R$. Then $1_R \in f^{-1}(P)$, and so $1_S = f(1_R) \in P$. This implies that $P = S$, which contradicts the fact that P is a prime ideal of S.

Suppose $a, b \in R$. Then

$$\begin{aligned} ab \in f^{-1}(P) &\implies f(ab) \in P \\ &\implies f(a)f(b) \in P && \text{(since } f \text{ is a ring homomorphism)} \\ &\implies f(a) \in P \text{ or } f(b) \in P && \text{(since } P \text{ is a prime ideal)} \\ &\implies a \in f^{-1}(P) \text{ or } b \in f^{-1}(P). \end{aligned}$$

Hence $f^{-1}(P)$ is a prime ideal of R.

QUESTION 4 **(3 pts)**. Let F be a field. Show that

$$I = \{a_0 + a_1 x + \cdots + a_n x^n : n \geq 0,\ a_0, \ldots, a_n \in F,\ a_0 + \cdots + a_n = 0\}$$

is an ideal of $F[x]$ and that $F[x]/I \cong F$ as rings.

Solution: Consider the evaluation homomorphism

$$\varphi_1\colon F[x] \to F, \quad \varphi(f(x)) = f(1).$$

For all $a \in F$, we have $\varphi_1(a) = a$, and so φ_1 is surjective. Since

$$\varphi_1(a_0 + a_1 x + \cdots + a_n x^n) = a_0 + \cdots + a_n,$$

we see that $\ker \varphi_1 = I$. Hence I is an ideal and, by the First Isomorphism Theorem, we have $F[x]/I \cong F$.

QUESTION 5 **(3 pts)**. Prove that $f(x) = x^4 + 4x^3 + 3x^2 + 7x + 10$ is irreducible in $\mathbb{Q}[x]$. *Hint*: Consider $f(x-1)$.

Solution: We have

$$\begin{aligned} f(x-1) &= (x-1)^4 + 4(x-1)^3 + 3(x-1)^2 + 7(x-1) + 10 \\ &= x^4 - 4x^3 + 6x^2 - 4x + 1 + 4(x^3 - 3x^2 + 3x - 1) + 3(x^2 - 2x + 1) + 7x - 7 + 10 \\ &= x^4 - 3x^2 + 9x + 3. \end{aligned}$$

Applying the Eisenstein criterion with $p = 3$, we see that $f(x-1)$ is irreducible. Thus $f(x)$ is also irreducible.

QUESTION 6 **(4 pts)**. In each case, give the number of elements of $F[x]/\langle f(x)\rangle$, and state whether or not $F[x]/\langle f(x)\rangle$ is a field.

(a) $f(x) = x^3 - x + 1$, $F = \mathbb{Z}_3$.

Solution: We see that $f(0) = 1$, $f(1) = 1$, and $f(-1) = 1$. Thus $f(x)$ has no roots in $\mathbb{Z}_3$. Since $\deg f(x) = 3$, this implies that $f(x)$ is irreducible. Hence the quotient $F[x]/\langle f(x)\rangle$ is a field. It has $3^3 = 27$ elements; these elements are $ax^2 + bx + c + \langle f(x)\rangle$ for $a, b, c \in \mathbb{Z}_3$.

(b) $f(x) = x^4 + x^3 + x + 1$, $F = \mathbb{Z}_2$.

Solution: Since $f(1) = 0$ in $\mathbb{Z}_2$, we see that $f(x)$ has a root and is therefore reducible. Thus $\mathbb{Z}_2[x]/\langle f(x)\rangle$ is not a field. It has $2^4 = 16$ elements; these elements are ax^3+bx^2+cx+d for $a, b, c, d \in \mathbb{Z}_2$.