MAT 3143 – Fall 2020

Final Exam

Professor: Alistair Savage

Your solutions should be submitted through Brightspace as a ***single pdf file****. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam ends at 12:30pm. You may not write anything on your pages after this time. You will then have until 12:40pm to scan and submit your solutions on Brightspace.

If you wish to leave the exam early, you must request permission in the Zoom chat. Once permission is granted, you may not write anything further on your pages, and you have 10 minutes to scan and submit your solutions.

Question 1 **(6 pts)**. For this question, you do *not* need to justify your answers.

(a) Give an example of a field F and a finitely-generated $F[x]$-module that is not finitely generated as an F-module.
(b) Give an example of a field F and reducible polynomial $f(x) \in F[x]$ with no roots in F.
(c) State the *structure theorem for finitely-generated modules over a principle ideal domain.*
(d) State the *modular irreducibility test* for polynomials.

Question 2 **(4 pts)**. Let I and J be ideals of a ring R.

(a) Prove that $I \cap J$ is an ideal of R.
(b) Is it necessarily true that $I \cup J$ is an ideal of R? Remember to justify your answer.

Question 3 **(4 pts)**.

(a) State the *Chinese Remainder Theorem.*
(b) Gemma is trying to pack chocolates into boxes. When he tries to put them evenly into three boxes, one is left over. When she tries to put them evenly into seven boxes, two are left over. What are the possibilities for the number of chocolates that Gemma has?

Question 4 **(4 pts)**. Prove that $\mathbb{R}[x]/\big(\mathbb{R}[x](x^2+1)\big) \cong \mathbb{C}$ as rings.

Question 5 **(3 pts)**.

(a) State the *Rational Roots Theorem.*
(b) Prove that $f(x) = x^3 + x^2 + 5x + 3$ is irreducible in $\mathbb{Q}[x]$.

Question 6 **(3 pts)**. Construct a field with 27 elements. Remember to justify that your ring is a field.

Question 7 **(4 pts)**.

(a) State the *ascending chain condition on principal ideals* (ACCP).
(b) Prove that every principal ideal domain satisfies the ACCP.

QUESTION 8 **(7 pts)**.

(a) Find all the units in $\mathbb{Z}(\sqrt{-7}) := \{a + b\sqrt{-7} : a, b \in \mathbb{Z}\}$.
(b) Show that $1 + \sqrt{-7}$ is an irreducible element of $\mathbb{Z}(\sqrt{-7})$ but is not prime.

QUESTION 9 **(4 pts)**.

(a) Suppose R is a ring. Define the *annihilator* $\operatorname{ann}(M)$ of an R-module M.
(b) Let V be a vector space over the field $\mathbb{R}$ of real numbers. Define the linear map

$$T\colon V \to V, \quad Tv = 2v,$$

and consider the associated $\mathbb{R}[x]$-module structure on V. Find a polynomial $p(x) \in \mathbb{R}[x]$ such that $\langle p(x) \rangle = \operatorname{ann}(V)$. Remember to justify your answer.

QUESTION 10 **(3 pts)**. Suppose M is a finite module over an infinite ring R. Prove that M is a free R-module if and only if $M = \{0\}$.

QUESTION 11 **(5 pts)**. Let R be an arbitrary ring.

(a) Define the *left regular module* of R.
(b) Let $s \in R$. Prove that the map

$$\rho_s\colon R \to R, \quad \rho_s(r) = rs.$$

is a homomorphism of R-modules.
(c) Prove that every homomorphism of R-modules $\varphi\colon R \to R$ is equal to ρ_s for a unique element $s \in R$.

QUESTION 12 **(3 pts)**. Suppose R is a commutative ring and $e \in R$ is an idempotent element. Furthermore, suppose that M is an R-module. For $r \in R$, define

$$rM := \{ru : u \in M\}.$$

Prove that $M = eM \oplus (1 - e)M$.