# University of Ottawa

Department of Mathematics and Statistics

MAT 3143: Ring Theory
Professor: Alistair Savage

Midterm Test
March 2, 2018

Surname ______________________ First Name ______________________

Student # ______________________

**Instructions:**

(a) You have 80 minutes to complete this exam.
(b) Unless otherwise indicated, you must justify your answers to receive full marks.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

| Question | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Maximum | 4 | 4 | 4 | 3 | 3 | 4 | 22 |
| Grade | | | | | | | |

QUESTION 1. [**4 points**] Suppose $R$ is a commutative ring.

(a) Give the definition of a *prime ideal*.

(b) For $a, b \in R$, write $a \mid b$ if $b = ra$ for some $r \in R$. Suppose $p \in R$ and $Rp \neq R$. Show that $Rp$ is a prime ideal if and only if, for all $a, b \in R$,

$$p \mid ab \implies (p \mid a \text{ or } p \mid b)\,.$$

QUESTION 2. [**4 points**] Suppose $S$ is a ring and let

$$R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; a, b, c \in S \right\}$$

be the upper triangular matrix ring over $S$, with usual matrix addition and multiplication. Show that

$$A = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \;\middle|\; a \in S \right\}$$

is an ideal of $R$ and prove that $R/A \cong S \times S$ (isomorphism of rings).

QUESTION 3. [**4 points**] Reagan is making loot bags for her birthday party and wants to distribute her Peppa Pig stickers evenly in the bags. She tries to put them evenly into three bags, but two are left out. She tries to put them evenly into four bags, but three are left out.

(a) What are all the possibilities for the number of Peppa Pig stickers that Reagan could have?

(b) Suppose Reagan tries to put the stickers evenly into six bags. Do you have enough information to know how many stickers will be left out? If so, how many will be left out?

QUESTION 4. [**3 points**]

(a) State the Rational Roots Theorem.

(b) Suppose $m, n \in \mathbb{Z}$, $n, m > 0$. Show that $\sqrt[n]{m}$ is not rational unless $m = k^n$ for some integer $k$.

QUESTION 5. [**3 points**]

(a) Prove that the polynomial $x^7 + 5x^5 - 20x^4 + 10x^2 - 15x + 30$ is irreducible in $\mathbb{Z}[x]$.

(b) Prove that the polynomial $25x^3 + 4x^2 + 17x + 1$ is irreducible in $\mathbb{Z}[x]$.

QUESTION 6. [**4 points**] Suppose $R$ is an integral domain and $a \in R$ is neither a unit nor a nilpotent. Prove that $R$ has infinitely many elements that are neither units nor nilpotents. In other words, find an infinite subset $A$ of $R$ such that, for every $b \in A$, $b$ is neither a unit nor a nilpotent.

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