University of Ottawa
Department of Mathematics and Statistics

MAT 3143: Ring Theory
Professor: Alistair Savage

Midterm Test – Solutions
March 2, 2018

Surname _____ First Name _____

Student # _____

Instructions:

(a) You have 80 minutes to complete this exam.
(b) Unless otherwise indicated, you must justify your answers to receive full marks.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	4	4	3	3	4	22
Grade							

QUESTION 1. **[4 points]** Suppose R is a commutative ring.

(a) Give the definition of a *prime ideal*.

(b) For $a, b \in R$, write $a \mid b$ if $b = ra$ for some $r \in R$. Suppose $p \in R$ and $Rp \neq R$. Show that Rp is a prime ideal if and only if, for all $a, b \in R$,

$$p \mid ab \implies (p \mid a \text{ or } p \mid b)\,.$$

Solution:

(a) An ideal P of a commutative ring R is prime if $P \neq R$ and

$$r, s \in R,\ rs \in P \implies r \in P \text{ or } s \in P.$$

(b) First note that, for $a, b \in R$, we have

$$a \mid b \iff \exists\, r \in R \text{ such that } b = ra \iff b \in Ra.$$

Now suppose Rp has the given property. Then

$$\begin{aligned} rs \in Rp &\implies p \mid rs \\ &\implies p \mid r \text{ or } p \mid s \\ &\implies r \in Rp \text{ or } s \in Rp. \end{aligned}$$

Hence Rp is prime.

Conversely, suppose Rp is prime. Then

$$\begin{aligned} p \mid ab &\implies ab \in Rp \\ &\implies a \in Rp \text{ or } b \in Rp \\ &\implies p \mid a \text{ or } p \mid b. \end{aligned}$$

QUESTION 2. **[4 points]** Suppose S is a ring and let

$$R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; a, b, c \in S \right\}$$

be the upper triangular matrix ring over S, with usual matrix addition and multiplication. Show that

$$A = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \;\middle|\; a \in S \right\}$$

is an ideal of R and prove that $R/A \cong S \times S$ (isomorphism of rings).

Solution: Consider the map

$$f \colon R \to S \times S, \quad \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mapsto (a, c).$$

Then, for $a_1, a_2, b_1, b_2, c_1, c_2 \in S$, we have

$$f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = (1, 1),$$

$$f\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\right) = (a_1 + a_2, c_1 + c_2) = f\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\right) + f\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\right),$$

$$f\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\right) = (a_1 a_2, c_1 c_2) = f\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\right) f\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\right).$$

Thus f is a ring homomorphism. Clearly $A = \ker f$, and so A is an ideal of R. Since f is obviously surjective, the First Isomorphism Theorem implies that

$$R/A \cong S \times S,$$

as desired.

QUESTION 3. **[4 points]** Reagan is making loot bags for her birthday party and wants to distribute her Peppa Pig stickers evenly in the bags. She tries to put them evenly into three bags, but two are left out. She tries to put them evenly into four bags, but three are left out.

(a) What are all the possibilities for the number of Peppa Pig stickers that Reagan could have?

(b) Suppose Reagan tries to put the stickers evenly into six bags. Do you have enough information to know how many stickers will be left out? If so, how many will be left out?

Solution: Let $x \in \mathbb{N}$ be the number of stickers. Then we have

$$\bar{x} = \bar{2} \text{ in } \mathbb{Z}_3,$$
$$\bar{x} = \bar{3} \text{ in } \mathbb{Z}_4.$$

Since $\gcd(3,4) = 1$, by the Chinese Remainder Theorem the map

$$\mathbb{Z}_{12} \to \mathbb{Z}_3 \times \mathbb{Z}_4, \quad x + 12\mathbb{Z} \mapsto (x + 3\mathbb{Z}, x + 4\mathbb{Z}),$$

is an isomorphism. We want to find a representative $m \in \{0, 1, 2, \ldots, 11\}$ such that

$$(m + 3\mathbb{Z}, m + 4\mathbb{Z}) = (m + 3\mathbb{Z}, m + 4\mathbb{Z}).$$

Comparing second components, we see that $m \in \{3, 7, 11\}$. Then, only $m = 11$ gives equality of the first components. Therefore, we know that $x \equiv 11 \mod 12$. Thus, the total number of stickers that Reagan has is of the form $11 + 12n$ for $n \in \mathbb{Z}$, $n \geq 0$.

Since $6 \mid 12$, we have a ring homomorphism

$$\mathbb{Z}_{12} \to \mathbb{Z}_6, \quad a + 12\mathbb{Z} \mapsto a + 6\mathbb{Z}.$$

It follows that $x \equiv 5 \mod 6$. So there will be five stickers left out if Reagan tries to put the stickers evenly into six bags. (So, yes, you have enough information to answer this question.)

QUESTION 4. **[3 points]**

(a) State the Rational Roots Theorem.

(b) Suppose $m, n \in \mathbb{Z}$, $n, m > 0$. Show that $\sqrt[n]{m}$ is not rational unless $m = k^n$ for some integer k.

Solution:

(a) Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ with $a_0 \neq 0$, $a_n \neq 0$. If $r \in \mathbb{Q}$ and $f(r) = 0$, then $r = \frac{c}{d}$ with $c \mid a_0$ and $d \mid a_n$.

(b) Suppose $\sqrt[n]{m}$ is rational. Then the polynomial $f(x) = x^n - m \in \mathbb{Z}[x]$ has a rational root. By the Rational Roots Theorem, any rational root of $f(x)$ is of the form $\pm k$ for some $k \mid m$. In particular, any rational root k satisfies

$$k \in \mathbb{Z},\ k^n = m.$$

QUESTION 5. **[3 points]**

(a) Prove that the polynomial $x^7 + 5x^5 - 20x^4 + 10x^2 - 15x + 30$ is irreducible in $\mathbb{Z}[x]$.

(b) Prove that the polynomial $25x^3 + 4x^2 + 17x + 1$ is irreducible in $\mathbb{Z}[x]$.

Solution:

(a) This follows from the Eisenstein criterion with $p = 5$.

(b) Reducing modulo 2 yields the polynomial $x^3 + x + 1$, which has no roots in $\mathbb{Z}_2$. Since it is cubic, this implies that is does not factor in $\mathbb{Z}_2$. Hence it cannot factor over $\mathbb{Z}$.

QUESTION 6. [**4 points**] Suppose R is an integral domain and $a \in R$ is neither a unit nor a nilpotent. Prove that R has infinitely many elements that are neither units nor nilpotents. In other words, find an infinite subset A of R such that, for every $b \in A$, b is neither a unit nor a nilpotent.

Solution: Consider the set

$$A = \{a^n \mid n \in \mathbb{Z},\ n \geq 1\}.$$

We first show that this set is infinite. Suppose that $a^n = a^m$ for some $n, m \in \mathbb{Z}$, $n, m \geq 1$. Without loss of generality, we assume $m \geq n$. Then we have

$$0 = a^m - a^n = a^n \left(a^{m-n} - 1\right).$$

Since R is a domain, this implies that $a^n = 0$ or $a^{m-n} - 1$. Since a is not nilpotent, we cannot have $a^n = 0$. Hence $a^{m-n} = 1$. If $m > n$, this implies that $aa^{m-n-1} = 1$, contradicting the fact that a is not a unit. Thus $m = n$. It follows immediately that the set A is infinite.

Now fix $n \in \mathbb{Z}$, $n \geq 1$. Suppose, towards a contradiction, that a^n is nilpotent. Then there exists some $m \geq 1$ such that

$$0 = (a^n)^m = a^{nm}.$$

But this contradicts the fact that a is not nilpotent. (In fact, it also contradicts the fact that R is an integral domain, hence has no zero divisors.)

Now suppose, again towards a contradiction, that a^n is a unit. Then there exists some $b \in R$ such that

$$1 = a^n b = a\left(a^{n-1}b\right) = \left(a^{n-1}b\right)a.$$

But this contradicts the fact that a is not a unit.

It follows that every element of the infinite set A is neither a unit nor a nilpotent.