University of Ottawa

Department of Mathematics and Statistics

MAT 3143: Ring Theory

Professor: Alistair Savage

Final Exam

April 2018

Surname ________________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) Unless otherwise indicated, you must justify your answers to receive full marks.
(c) Write your student number at the top of each page in the space provided.
(d) You may use the last page of the exam as scrap paper.

Mobile phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have read and ensured that you are complying with the above statement.

Signature: ____

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	11	12	Total
Maximum	3	3	3	3	3	4	4	6	7	5	3	3	47
Grade													

QUESTION 1. Give an example of each of the following. You should justify that your example satisfies the given requirements.

(a) [**1 point**] An idempotent element of the ring $M_n(\mathbb{R})$ of $n \times n$ real matrices, $n \geq 2$, that is neither the zero matrix nor the identity matrix. Do *not* choose a specific value for n. (That is, give an answer for each $n \geq 2$.)

(b) [**1 point**] A field extension of $\mathbb{Q}$ of degree n, $n \geq 2$. Do *not* choose a specific value for n. (That is, give an answer for each $n \geq 2$.)

(c) [**1 point**] A pair of rings $R \subseteq S$ and an element a that is irreducible in S but is not irreducible in R.

QUESTION 2. [**3 points**] Suppose that $\theta\colon R \to S$ is a ring homomorphism such that $\theta(R)$ and $\ker\theta$ both contain no nonzero nilpotent elements. Show that R contains no nonzero nilpotent elements.

QUESTION 3. Consider the polynomial

$$f(x) = x^3 + 16x + 6.$$

(a) [**1 point**] Is $f(x)$ irreducible over $\mathbb{Q}$? Remember to justify your answer.

(b) [**1 point**] Is $f(x)$ irreducible over $\mathbb{Z}_3$? Remember to justify your answer.

(c) [**1 point**] Is $f(x)$ irreducible over $\mathbb{Z}_5$? Remember to justify your answer.

QUESTION 4. [**3 points**] Let

$$A = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid n \in \mathbb{N},\ a_0 \in 5\mathbb{Z},\ a_1, \ldots, a_n \in \mathbb{Z}\}$$

Prove that A is an ideal of $\mathbb{Z}[x]$ and that $\mathbb{Z}[x]/A \cong \mathbb{Z}_5$ (isomorphism of rings).

QUESTION 5. [**3 points**] Suppose R is an integral domain. Prove that if $R[x]$ is a PID, then R is a field. *Hint*: Suppose $R[x]$ is a PID and choose a nonzero element $a \in R$. Use the fact that the ideal $\langle a, x \rangle$ is principal to show that a is a unit.

QUESTION 6. Let $p = 2 + \sqrt{-5} \in \mathbb{Z}(\sqrt{-5})$.

(a) [**2 points**] Show that p is irreducible in $\mathbb{Z}(\sqrt{-5})$.

(b) [**2 points**] Show that p is not prime in $\mathbb{Z}(\sqrt{-5})$.

QUESTION 7.

(a) [**1 point**] Give the definition of a *euclidean domain*.

(b) [**3 points**] Prove that every euclidean domain is a PID.

QUESTION 8.

(a) [**2 points**] Let R be a ring. Suppose that

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

is an infinite chain of ideals of R. Prove that $\bigcup_{n=1}^{\infty} I_n$ is an ideal of R.

(b) [**1 point**] If I and J are ideals of a ring R, it is necessarily true that $I \cup J$ is an ideal of R? Remember to justify your answer.

(c) [**1 point**] State the ACCP for integral domains.

(d) [**2 points**] Prove that every PID satisfies the ACCP.

QUESTION 9. Suppose $F \subseteq E$ is a field extension and let $u, v \in E$ be algebraic over F of degrees m, n (respectively).

(a) [**2 points**] Show that $[F(u,v) : F] \leq mn$.

(b) [**2 points**] Show that, if m and n are relatively prime, then $[F(u,v) : F] = mn$.

(c) [**3 points**] Is the converse to (b) true? Justify your answer.

QUESTION 10. Consider the polynomial $f(x) = x^3 - 7$.

(a) [**1 point**] Find the splitting field E of $f(x)$ over $\mathbb{Q}$.

(b) [**4 points**] What is $[E : \mathbb{Q}]$? Give a basis for E over $\mathbb{Q}$.

QUESTION 11. Suppose $F \subseteq E$ is a field extension and $u, v \in E$.

(a) [**1 point**] Show that if $u - v \in F$, then $F(u) = F(v)$.

(b) [**2 points**] If $u^2 - v^2 \in F$, it is necessarily true that $F(u) = F(v)$? Remember to justify your answer.

QUESTION 12.

(a) **[1 point]** Prove that $f(x) = x^3 + x^2 + x + 3 \in \mathbb{Z}_5[x]$ is irreducible.

(b) **[2 points]** Construct a field of order 125.

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