1 $\quad S = \{ y \in R \mid yx = xy \quad \forall x \in X \}$.

(a) $0_R, 1_R \in S$.

$$0 \cdot x = 0 = x \cdot 0 \quad \forall x \in X$$

$$1 \cdot x = x = x \cdot 1 \quad \forall x \in X$$

(b). $(y_1 + y_2)x = y_1 x + y_2 x = x y_1 + x y_2 =$

$= x(y_1 + y_2) \quad \forall x \in X \quad \forall y_1, y_2 \in S$

(c) $y_1(y_2 x) = (y_1 x) y_2 = x y_1 y_2 \quad \forall x \quad \forall y_1, y_2 \in S$

2 $M_2(\mathbb{Z})$ is NOT a domain as it has a zero divisor

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(3) Take

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; a,b,c \in \mathbb{Z} \right\}.$$

and $$I = \left\{ \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \;\middle|\; d \in \mathbb{Z} \right\}.$$

$$R/I = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \;\middle|\; x,y \in \mathbb{Z} \right\}.$$

Check that R is a non-com. ring.

I is an ideal

R/I is a commutative ring

(4) Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a ring hom

then $\varphi(1+i^2) = \varphi(1) + \varphi(i^2) =$

$0 = \varphi(0)$ $= \varphi(1) + \varphi(i)^2$

So $1 + \varphi(i)^2 = 0$ in $\mathbb{R}$, a contradiction

(5) $h(x) = x(x-1)$.

We claim $(h) \subset \{f \in \mathbb{Z}_5[x] \mid f(0) = f(1) = 0\}$

Indeed. $(h) = \{h(x)g(x) \mid \forall g \in \mathbb{Z}_5[x]\}$.

$f = h \cdot g$ $\quad h(0)g(0) = 0$

$h(1)g(1) = 0$

$(h) \supset \{f \in \mathbb{Z}_5[x] \mid f(0) = f(1) = 0\}$.

Indeed, in $\mathbb{Z}_5[x]$ $\quad f(x) = x(x-1)g(x)$ for some $g \in \mathbb{Z}_5[x]$ (0 and 1 are roots of f).

6

Consider $I_1 = (x)$ $\qquad R = F[x]$.

$I_2 = (x-1)$

Then $I_1 + I_2 = R$ are coprime

So by the Chinese R. Thm

$$\begin{array}{ccccc} R/I_1 I_2 & \simeq & R/I_1 & \times & R/I_2 \\ \wr\!\wr & & \wr\!\wr & & \wr\!\wr \\ F[x]/(x^2-x) & & F & & F. \end{array}$$