University of Ottawa

Department of Mathematics and Statistics

MAT 3143 : Ring Theory

Professor : Hadi Salmasian

Midterm Exam

February 27, 2015

Surname ______________________ First Name ______________________

Student # ______________________

Instructions :

(a) You have 80 minutes to complete this exam.

(b) The number of points available for each question is indicated in square brackets.

(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(d) Write your student number at the top of each page in the space provided.

(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(f) You are strongly recommended to write in **pen**, not pencil.

(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	3	6	7	8	6	5	35
Grade							

1. **[3 points]** Prove that every finite integral domain is a field.

2. **[6 points]** Prove the Rational Roots Theorem :
Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ be a non-constant polynomial, and $r \in \mathbb{Q}$ be such that $f(r) = 0$. Prove that there exist $p, q \in \mathbb{Z}$ such that $p|a_0$, $q|a_n$, and $r = \dfrac{p}{q}$.

3. **[7 points]** Factor the polynomial $f(x) = 2x^4 + x^3 + 2x^2 + 3x + 1$ in $\mathbb{Q}[x]$ as much as possible. You should justify that the factorization that you obtain cannot be refined.

4. (a) **[2 points]** Write down the definition of a prime ideal.

   (b) **[4 [points]** Let $\mathbb{F}$ and $\mathbb{K}$ be two fields. Show that every non-zero proper ideal of the ring $M_2(\mathbb{F}) \times M_2(\mathbb{K})$ is a maximal ideal.

   (c) **[2 points]** Let $\mathbb{F}$ be a field. Give an example of a non-zero proper ideal of the ring $M_2(\mathbb{F}) \times M_2(\mathbb{F}) \times M_2(\mathbb{F})$ which is not a maximal ideal. You should justify your answer.

5. (a) **[2 points]** Write down the definition of the characteristic of a ring.

   (b) **[4 points]** Let R be a commutative ring such that $\mathrm{char}(R) = 4$. Prove that the map $\phi : R[x] \to R$ defined by

   $$\phi(a_0 + a_1 x + \cdots + a_n x^n) = a_0 + 2a_1$$

   is a ring homomorphism. Is $\ker(\phi)$ a maximal ideal? You should justify your answer.

   Hint : You can use the Evaluation Theorem.

6. Let R be a commutative ring.

(a) **[2 points]** Suppose that $a, b \in R$ satisfy the relations $a^2 = a$ and $2ab = b$. Prove that $b = 0$.

(b) **[3 points]** Let $f(x) \in R[x]$ be an idempotent element of $R[x]$. Prove that $f(x)$ is a constant polynomial.

Hint : You can start as follows. Suppose that $f(x) = a_0 + \cdots + a_n x^n$ with $n \geq 1$ and $a_n \neq 0$. Assume that $r \geq 1$ is the smallest positive integer satisfying $a_r \neq 0$, and write $f(x)$ in the form

$$f(x) = a_0 + a_r x^r + \cdots + a_n x^n = a_0 + a_r x^r + x^{r+1} g(x) \text{ where } g(x) \in R[x].$$

This page is intentionally left blank. You may use it as scrap paper.