University of Ottawa

Department of Mathematics and Statistics

MAT 3143 : Ring Theory

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Midterm Exam

February 27, 2015

Surname ______________________ First Name ______________________

Student # ______________________

Instructions :

(a) You have 80 minutes to complete this exam.

(b) The number of points available for each question is indicated in square brackets.

(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(d) Write your student number at the top of each page in the space provided.

(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(f) You are strongly recommended to write in **pen**, not pencil.

(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	3	6	7	8	6	5	35
Grade							

1. **[3 points]** Prove that every finite integral domain is a field.

   Solution: This is Theorem 1.47 of the notes.

2. **[6 points]** Prove the Rational Roots Theorem :
Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ be a non-constant polynomial, and $r \in \mathbb{Q}$ be such that $f(r) = 0$. Prove that there exist $p, q \in \mathbb{Z}$ such that $p | a_0$, $q | a_n$, and $r = \frac{p}{q}$.

Solution: This is Theorem 2.17 of the notes.

3. **[7 points]** Factor the polynomial $f(x) = 2x^4 + x^3 + 2x^2 + 3x + 1$ in $\mathbb{Q}[x]$ as much as possible. You should justify that the factorization that you obtain cannot be refined.

Solution: We start by seeking a rational root for $f(x)$. By the Rational Root Theorem, we should look for such roots in the set

$$\left\{\frac{1}{2}, -\frac{1}{2}, 1, -1\right\}.$$

We then obtain

$$f(-\frac{1}{2}) = \frac{1}{8} - \frac{1}{8} + \frac{1}{2} - \frac{3}{2} + 1 = 0.$$

By the long division algorithm we obtain

$$\begin{aligned} f(x) &= (x + \frac{1}{2})2x^3 + (2x^2 + 3x + 1) \\ &= (x + \frac{1}{2})2x^3 + (x + \frac{1}{2})(2x) + (2x + 1) \\ &= (x + \frac{1}{2})(2x^3 + 2x + 2) = (2x + 1)(x^3 + x + 1). \end{aligned}$$

Now since $\deg(x^3 + x + 1) \leq 3$, by Theorem 2.25, $x^3 + x + 1$ is reducible in $\mathbb{Q}[x]$ if and only if it has a root in $\mathbb{Q}$. But again the Rational Root Theorem implies that the roots of $x^3 + x + 1$ in $\mathbb{Q}$ belong to the set $\{1, -1\}$. Since none of them is a root of $x^3 + x + 1$, the latter polynomial is irreducible and the factorization cannot be refined any further.

4. (a) **[2 points]** Write down the definition of a prime ideal.

Solution: An ideal I of a commutative ring R is called a prime ideal if

$$\forall_{a,b\in R}\, ab \in I \;\Rightarrow\; a \in I \text{ or } b \in I.$$

(b) **[4 [points]** Let $\mathbb{F}$ and $\mathbb{K}$ be two fields. Show that every non-zero proper ideal of the ring $M_2(\mathbb{F}) \times M_2(\mathbb{K})$ is a maximal ideal.

Solution: By Proposition 1.89, every ideal of $R \times S$ is of the form $A \times B$ where $A \subseteq R$ and $B \subseteq S$ are arbitrary ideals. But Theorem 1.72 implies that $M_2(\mathbb{F})$ and $M_2(\mathbb{K})$ are simple rings. Therefore the only ideals of $M_2(\mathbb{F}) \times M_2(\mathbb{K})$ are

$$\{0\} \times \{0\}\,,\; I = M_2(\mathbb{F}) \times \{0\}\,,\; J = \{0\} \times M_2(\mathbb{K})\,,\; \text{ and } M_2(\mathbb{F}) \times M_2(\mathbb{K}).$$

Now set $R = M_2(\mathbb{F}) \times M_2(\mathbb{K})$. Then$R/I \cong M_2(\mathbb{K})$ is a simple ring, and therefore Theorem 1.73 implies that I is a maximal ideal. A similar argument proves that J is also a maximal ideal.

(c) **[2 points]** Let $\mathbb{F}$ be a field. Give an example of a non-zero proper ideal of the ring $M_2(\mathbb{F}) \times M_2(\mathbb{F}) \times M_2(\mathbb{F})$ which is not a maximal ideal. You should justify your answer.

Solution: The ideal $M_2(\mathbb{F}) \times \{0\} \times \{0\}$ is not maximal because

$$M_2(\mathbb{F}) \times \{0\} \times \{0\} \subsetneq M_2(\mathbb{F}) \times M_2(\mathbb{F}) \times \{0\} \subsetneq R$$

5. (a) **[2 points]** Write down the definition of the characteristic of a ring.

Solution: Given a ring R, the smallest integer $n > 0$ such that

$$\underbrace{1 + \cdots + 1}_{n \text{ times}} = 0$$

is called the characteristic of R. If no such integer n exists, then we define $\operatorname{char}(R) = 0$.

(b) **[4 points]** Let R be a commutative ring such that $\operatorname{char}(R) = 4$. Prove that the map $\phi : R[x] \to R$ defined by

$$\phi(a_0 + a_1 x + \cdots + a_n x^n) = a_0 + 2a_1$$

is a ring homomorphism. Is $\ker(\phi)$ a maximal ideal? You should justify your answer.

Hint : You can use the Evaluation Theorem.

Solution: Note that $2^n = 0$ in R for every $n \geq 2$. Therefore $\phi(f(x)) = f(2)$, and by the Evaluation Theorem, ϕ is a surjective ring homomorphism. Now by the Isomorphism Theorem,

$$R[x]/\ker(\phi) \cong R.$$

If $\ker(\phi)$ is a maximal ideal, then $R[x]/\ker(\phi)$ should be a field (Corollary 1.74), and therefore R should be a field. But this is a contradiction because the characteristic of a field is always a prime number, whereas $\operatorname{char}(R) = 4$ is not prime.

6. Let R be a commutative ring.

(a) **[2 points]** Suppose that $a, b \in R$ satisfy the relations $a^2 = a$ and $2ab = b$. Prove that $b = 0$.

Solution:

$$2ab = b \;\Rightarrow\; 2ab = 2a^2b = a(2ab) = ab \;\Rightarrow\; ab = 0 \;\Rightarrow\; b = 2ab = 0.$$

(b) **[3 points]** Let $f(x) \in R[x]$ be an idempotent element of $R[x]$. Prove that $f(x)$ is a constant polynomial.

Hint : You can start as follows. Suppose that $f(x) = a_0 + \cdots + a_n x^n$ with $n \geq 1$ and $a_n \neq 0$. Assume that $r \geq 1$ is the smallest positive integer satisfying $a_r \neq 0$, and write $f(x)$ in the form

$$f(x) = a_0 + a_r x^r + \cdots + a_n x^n = a_0 + a_r x^r + x^{r+1} g(x) \text{ where } g(x) \in R[x].$$

Solution: Using the hint, we obtain $f(x) = a_0 + a_r x^r + x^{r+1} g(x)$ and therefore $f(x)^2 = a_0^2 + 2a_0 a_r x^r + x^{r+1} h(x)$ for some $h(x) \in R[x]$. From the equality $f(x) = f(x)^2$ it follows that $a_0 = a_0^2$ and $a_r = 2a_0 a_r$. Now from part (a) for $a = a_0$ and $b = a_r$ we obtain $a_r = 0$, which is a contradiction.