MAT 3143 – Winter 2013

Midterm Test

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Unless otherwise indicated, you must justify your answers in order to receive full marks.

Question 1. **(5 points)** Give an example of each of the following. No justification is required.

(a) A noncommutative division ring.
(b) A prime ideal that is not a maximal ideal.
(c) A general ring that is not a ring.
(d) A field with 4 elements.
(e) An idempotent element in $M_2(\mathbb{Z})$, the ring of 2×2 matrices with integer entries, that is neither the identity element nor the zero element.

Question 2. **(4 points)** Suppose that $\mathbb{F}$ is a field and $a \in \mathbb{F}$. Prove that $\mathbb{F}[x]/\langle x - a \rangle$ is isomorphic (as a ring) to $\mathbb{F}$.

Question 3. **(3 points)** Show that a general ring homomorphism $\theta \colon \mathbb{Z} \to \mathbb{Z}$ is either a ring homomorphism or $\theta(k) = 0$ for all $k \in \mathbb{Z}$.

Question 4. **(2 points)** Show that there are infinitely many integers n for which $x^8 + 30x^3 - 55x + n$ is irreducible in $\mathbb{Q}[x]$.

Question 5. **(6 points)**

(a) Show that $\mathbb{Z}_4[x]$ has infinitely many units and infinitely many nilpotent elements.
(b) Find a polynomial in $\mathbb{Z}_4[x]$ that is neither a unit nor nilpotent.

Question 6. **(4 points)**

(a) Find the (monic) greatest common divisor of $x^3 + \bar{1}$ and $x^3 + x^2 + x + \bar{1}$ in $\mathbb{Z}_3[x]$.
(b) Factor $x^3 + x^2 + x + \bar{1}$ as a product of irreducible polynomials in $\mathbb{Z}_3[x]$.