MAT 3143 – Winter 2013

Midterm Test – Solutions

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Unless otherwise indicated, you must justify your answers in order to receive full marks.

QUESTION 1. **(5 points)** Give an example of each of the following. No justification is required.

(a) A noncommutative division ring.

Solution: The quaternions are one example.

(b) A prime ideal that is not maximal ideal.

Solution: In any nonzero integral domain that is not a field (e.g. $\mathbb{Z}$), the zero ideal is prime but not maximal. (In a field, the zero ideal is both prime and maximal.)

(c) A general ring that is not a ring.

Solution: The general ring $2\mathbb{Z}$ is one example.

(d) A field with 4 elements.

Solution: An example is given by $\mathbb{Z}_2[x]/\langle h(x)\rangle$ for any irreducible polynomial $h(x) \in \mathbb{Z}_2[x]$ of degree two. For instance, one can take $h(x) = x^2 + x + \bar{1}$. Since $h(x)$ has no roots in $\mathbb{Z}_2$ (and has degree less than or equal to three), is it irreducible.

(e) An idempotent element in $M_2(\mathbb{Z})$, the ring of 2×2 matrices with integer entries, that is neither the identity element nor the zero element.

Solution: One example is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

QUESTION 2. **(4 points)** Suppose that $\mathbb{F}$ is a field and $a \in \mathbb{F}$. Prove that $\mathbb{F}[x]/\langle x - a\rangle$ is isomorphic (as a ring) to $\mathbb{F}$.

Solution: Since $\mathbb{F}$ is commutative (hence $a \in Z(\mathbb{F})$), we have the evaluation ring homomorphism $\varphi_a \colon \mathbb{F}[x] \to \mathbb{F}$, $\varphi(f(x)) = f(a)$. This map is surjective since, for example, $\varphi(b) = b$ for all $b \in \mathbb{F}$. Now,

$$\begin{aligned} \ker \varphi_a &= \{f(x) \in \mathbb{F}[x] \mid f(a) = 0\} \\ &= \{f(x) \in \mathbb{F}[x] \mid (x - a)|f(x)\} \qquad \text{(by the Factor Theorem)} \\ &= \langle x - a\rangle. \end{aligned}$$

Therefore, by the First Isomorphism Theorem, we have $\mathbb{F}[x]/\langle x - a\rangle \cong \mathbb{F}$.

QUESTION 3. **(3 points)** Show that a general ring homomorphism $\theta \colon \mathbb{Z} \to \mathbb{Z}$ is either a ring homomorphism or $\theta(k) = 0$ for all $k \in \mathbb{Z}$.

Solution: Suppose $\theta \colon \mathbb{Z} \to \mathbb{Z}$ is a general ring homomorphism and let $a = \theta(1)$. Then $\theta(2) = \theta(1 + 1) = \theta(1) + \theta(1) = a + a = 2a$. Thus

$$4a = a+a+a+a = \theta(1)+\theta(1)+\theta(1)+\theta(1) = \theta(1+1+1+1) = \theta(4) = \theta(2\cdot 2) = \theta(2)\theta(2) = (2a)(2a) = 4a^2.$$

Therefore $a^2 = a$ and so a is equal to 0 or 1. If $a = 0$, then, for all $k \in \mathbb{Z}$, we have $\theta(k) = \theta(k1) = k\theta(1) = k0 = 0$. On the other hand, if $a = 1$, then, for all $k \in \mathbb{Z}$, we have $\theta(k) = \theta(k1) = k\theta(1) = k1 = k$. Hence θ is a ring homomorphism (in fact, it is the identity map).

QUESTION 4. **(2 points)** Show that there are infinitely many integers n for which $x^8 + 30x^3 - 55x + n$ is irreducible in $\mathbb{Q}[x]$.

Solution: Suppose $n = 3^k \cdot 5$ for $k \in \mathbb{N}$. (One could replace 3 by any prime other than 5 here.) Then 5 divides n, but 5^2 does not. Applying the Eisenstein Criterion with the prime 5, we see that $f(x)$ is irreducible in $\mathbb{Q}[x]$. Since there are an infinite number of choices of k, we are done.

QUESTION 5. **(6 points)**

(a) Show that $\mathbb{Z}_4[x]$ has infinitely many units and infinitely many nilpotent elements.

Solution: For all $k \in \mathbb{N}$, the element $\bar{2}x^k \in \mathbb{Z}_4[x]$ is nilpotent since $(\bar{2}x^k)^2 = \bar{0}$. Thus $\mathbb{Z}_4[x]$ has infinitely many nilpotent elements. Also, for $k \in \mathbb{N}$, the element $\bar{2}x^k + \bar{3} \in \mathbb{Z}_4[x]$ is a unit since $(\bar{2}x^k + \bar{3})^2 = \bar{1}$. Thus $\mathbb{Z}_4[x]$ has infinitely many units.

(b) Find a polynomial in $\mathbb{Z}_4[x]$ that is neither a unit nor nilpotent.

Solution: The polynomial $x \in \mathbb{Z}_4[x]$ is not a unit since for any nonzero polynomial $f(x) \in \mathbb{Z}_4[x]$, we have $\deg xf(x) = \deg x + \deg f(x) \geq \deg x = 1$ by Theorem 2.1.9(f) of the lecture notes (since the leading coefficient of x is a unit in $\mathbb{Z}_4$) and thus $xf(x) \neq 1$. Furthermore, for all positive integers n, we have $x^n \neq 0$ and so x is not nilpotent.

QUESTION 6. **(4 points)**

(a) Find the (monic) greatest common divisor of $x^3 + \bar{1}$ and $x^3 + x^2 + x + \bar{1}$ in $\mathbb{Z}_3[x]$.

Solution: We use the Euclidean Algorithm. We have

$$\begin{aligned} x^3 + x^2 + x + \bar{1} &= (x^3 + \bar{1}) + (x^2 + x) \\ x^3 + \bar{1} &= (x - 1)(x^2 + x) + (x + \bar{1}) \\ x^2 + x &= x(x + \bar{1}) \end{aligned}$$

Thus, $\gcd(x^3 + \bar{1}, x^3 + x + \bar{1}) = x + \bar{1}$.

(b) Factor $x^3 + x^2 + x + \bar{1}$ as a product of irreducible polynomials in $\mathbb{Z}_3[x]$.

Solution: By the above, we know that $x + \bar{1}$ is a factor of $x^3 + x^2 + x + \bar{1}$. Polynomial division then gives $x^3 + x^2 + x + \bar{1} = (x + \bar{1})(x^2 + \bar{1})$. Since $x^2 + \bar{1}$ has no roots in $\mathbb{Z}_3$, it is irreducible. So we are done.