University of Ottawa

Department of Mathematics and Statistics

MAT 3143: Ring Theory
Professor: Hadi Salmasian

Midterm Exam

March 1, 2011

Surname ________________________ First Name ________________________

Student # ________________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You are strongly recommended to write in **pen**, not pencil.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	11	12	Total
Maximum	4	4	4	5	4	4	4	5	6	4	4	2	50
Grade													

1.

(a) [**2 pts**] State the definition of a prime ideal.

(b) [**2 pts**] State the definition of a maximal ideal.

(c) [**1 pts**] Give an example of a prime ideal in the ring $\mathbb{C}[x]$ which is not a maximal ideal.

(d) [**4 pts**] Prove the following theorem:

Let R be a commutative ring and $P \subsetneq R$ be an ideal of R. Then P is a prime ideal if and only if R/P is an integral domain.

2.

(a) **[2 pts]** State the definition of an irreducible polynomial in $\mathbb{F}[x]$, where $\mathbb{F}$ is an arbitrary field.

(b) **[4 pts]** Prove the Rational Roots Theorem:

Let $n \geq 1$ and $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial in $\mathbb{Z}[x]$. Assume that $a_0 \neq 0$ and $a_n \neq 0$. Then every rational root of $f(x)$ has the form $\frac{c}{d}$ where $c|a_0$ and $d|a_n$.

(c) **[3 pts]** Factor the polynomial $2x^4 + x^3 + 6x + 3$ in $\mathbb{Q}[x]$ as much as possible. You should justify your answer, i.e., write your solution in detail. (Only writing the final answer is not acceptable.)

3. Which of the following statements are true? Which of them are false? You should justify your answers, i.e., you should prove or disprove each statement.

(a) [**1 pt**] The polynomial $4x^7 - 3x^2 + 6x - 12$ is irreducible in $\mathbb{Q}[x]$.

(b) [**1 pt**] The ring $\mathbb{Z}_2 \times \mathbb{C}$ is a division ring.

(c) [**1 pt**] If $\theta : R \to S$ is a surjective ring homomorphism, then $\theta(Z(R)) \subseteq Z(S)$.

(d) **[1 pt]** The polynomial $x^{100} + 10x^{50} - 5x^{25} + 15$ is irreducible in $\mathbb{R}[x]$.

(e) **[1 pt]** There exists a polynomial $f(x) \in \mathbb{Q}[x]$ such that $\deg(f(x)) = 2011$ and $f(x)$ is irreducible in $\mathbb{Q}[x]$.

4. In each of the following, give an appropriate example. You do not need to justify your example.

(a) [**1 pt**] Give an example of a nonzero nilpotent element in the ring $\mathrm{M}_2(\mathbb{R})$. (Recall that $\mathrm{M}_2(\mathbb{R})$ is the ring of 2×2 matrices with real entries.)

(b) [**1 pt**] Give an example of a nonzero proper ideal of the ring $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

(c) [**1 pt**] Give an example of a ring homomorphism $\theta : \mathbb{C} \to \mathbb{C}$ other than the identity map.

(d) [**1 pt**] Give an example of a subring of $\mathbb{Z} \times \mathbb{Z}$ other than $\{(0,0)\}$ and $\mathbb{Z} \times \mathbb{Z}$.

(e) [**1 pt**] Give an example of an infinite ring R such that $\mathrm{char}(R) = 3$.

5. [**2 pts**] Let R be a ring. Prove that the map

$$\theta : R[x] \to R$$

defined by

$$\theta(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n) = a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$$

is a ring homomorphism. What is $\ker(\theta)$?

6. **(Bonus Question)**
   (a) [**1 pt**] Let R be a commutative ring. Prove that if two elements $r, s \in R$ are nilpotent, then $r + s$ is also nilpotent.

   (b) [**1 pt**] By an example, show that the statement of part (a) is not true if R is not assumed to be commutative.

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