University of Ottawa

Department of Mathematics and Statistics

MAT 3143: Ring Theory

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Midterm Exam

March 1, 2011

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You are strongly recommended to write in **pen**, not pencil.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	11	12	Total
Maximum	4	4	4	5	4	4	4	5	6	4	4	2	50
Grade													

1.

(a) **[2 pts]** State the definition of a prime ideal.

Solution: An ideal P of a commutative ring R is called a prime ideal iff P is proper and satisfies the following:

$$\text{for every } a, b \in R, \quad \text{if } ab \in P \text{ then } a \in P \text{ or } b \in P.$$

(b) **[2 pts]** State the definition of a maximal ideal.

Solution: An ideal A of a ring R is called maximal iff A is proper and there is no ideal B of R such that $A \subsetneq B \subsetneq R$.

(c) **[1 pts]** Give an example of a prime ideal in the ring $\mathbb{C}[x]$ which is not a maximal ideal.

Solution: Since $\mathbb{C}[x]$ is an integral domain, the zero ideal $\{0\}$ is a prime ideal. However, it is not maximal because for example $\{0\} \subsetneq \langle x \rangle \subsetneq R$.

(d) **[4 pts]** Prove the following theorem:

Let R be a commutative ring and $P \subsetneq R$ be an ideal of R. Then P is a prime ideal if and only if R/P is an integral domain.

Solution: First assume P is a prime ideal. Since R is commutative, we know that R/P is commutative as well. Suppose $P + a \in R/P$ and $P + b \in R/P$ are elements such that $(P + a)(P + b) = P + 0$. Then

$$\begin{aligned} P + 0 = (P + a)(P + b) = P + ab \Rightarrow ab \in P \Rightarrow a \in P \text{ or } b \in P \\ \Rightarrow P + a = P + 0 \text{ or } P + b = P + 0 \end{aligned}$$

which shows that the ring R/P is an integral domain.

Conversely, assume R/P is an integral domain. If $a, b \in R$ and $ab \in P$ then we have:

$$\begin{aligned} ab \in P \Rightarrow (P + a)(P + b) = P + ab = P + 0 \\ \Rightarrow P + a = P + 0 \text{ or } P + b = P + 0 \Rightarrow a \in P \text{ or } b \in P \end{aligned}$$

which shows that P is a prime ideal of R.

2.

(a) **[2 pts]** State the definition of an irreducible polynomial in $\mathbb{F}[x]$, where $\mathbb{F}$ is an arbitrary field.

Solution: A polynomial $f(x) \in \mathbb{F}[x]$ is called irreducible iff:

- $f(x) \neq 0$ and $\deg(f(x)) \geq 1$.
- If $f(x) = g(x)h(x)$ for $g(x), h(x) \in \mathbb{F}[x]$, then either $\deg(g(x)) = 0$ or $\deg(h(x)) = 0$.

(b) **[4 pts]** Prove the Rational Roots Theorem:

Let $n \geq 1$ and $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial in $\mathbb{Z}[x]$. Assume that $a_0 \neq 0$ and $a_n \neq 0$. Then every rational root of $f(x)$ has the form $\dfrac{c}{d}$ where $c|a_0$ and $d|a_n$.

Solution: Let $r = \frac{c}{d}$ be a rational root of $f(x)$ written in lowest terms. Then

$$f(\frac{c}{d}) = 0 \Rightarrow a_0 d^n + a_1 d^{n-1} c \cdots + a_n c^n = 0 \Rightarrow a_0 d^n = -(a_1 d^{n-1} c + \cdots + a_n c^n)$$

The right hand side is divisible by c, therefore the left hand side should also be divisible by c. But c and d are relatively prime. Therefore from $c|d^n a_0$ it follows that $c|a_0$.

Similarly, we can write $a_n c^n = -(a_{n-1} c^{n-1} d + \cdots + a_0 d^n)$, the right hand side is divisible by d, therefore we should have $d|a_n c^n$. Since c and d are relatively prime it follows that $d|a_n$.

(c) **[3 pts]** Factor the polynomial $2x^4 + x^3 + 6x + 3$ in $\mathbb{Q}[x]$ as much as possible. You should justify your answer, i.e., write your solution in detail. (Only writing the final answer is not acceptable.)

Solution: By the Rational Root Theorem the possible rational roots of this polynomial are

$$\{\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}\}.$$

A simple calculation shows that $-\frac{1}{2}$ is a root. We now divide $2x^4 + x^3 + 6x + 3$ by $x + \frac{1}{2}$:

$$\begin{array}{rlll}
x + \frac{1}{2} & | \ \ 2x^3 + 6 & & \\
\cline{2-4}
 & 2x^4 + x^3 & +6x + 3 & \\
- & 2x^4 + x^3 & & \\
\cline{2-4}
 & \qquad\qquad 0 & +6x + 3 & \\
 & \qquad\qquad - & 6x + 3 & \\
\cline{3-4}
 & & \qquad 0 &
\end{array}$$

Which means that:
$2x^4 + x^3 + 6x + 3 = (x + \frac{1}{2})(2x^3 + 6) = (2x + 1)(x^3 + 3)$.

Now $\deg(x^3 + 3) = 3$, and therefore by a theorem mentioned in class, if this polynomial is reducible in $\mathbb{Q}[x]$ then it should also have a root in $\mathbb{Q}$. However, again by the Rational Root Theorem the only possibilities for a rational root are $\{\pm 1, \pm 3\}$, and none of these numbers is a root of $x^3 + 3$. Therefore the latter polynomial is irreducible in $\mathbb{Q}[x]$ and the factorization obtained above cannot be factored any further.

3. Which of the following statements are true? Which of them are false? You should justify your answers, i.e., you should prove or disprove each statement.

(a) [**1 pt**] The polynomial $4x^7 - 3x^2 + 6x - 12$ is irreducible in $\mathbb{Q}[x]$.

Solution: It is true. The Eisenstein Criterion for $p = 3$ is applicable.

(b) [**1 pt**] The ring $\mathbb{Z}_2 \times \mathbb{C}$ is a division ring.

Solution: It is false: this ring contains nonzero elements which are not invertible. For example, $(\overline{0}, 1) \in \mathbb{Z}_2 \times \mathbb{C}$ is not invertible.

(c) [**1 pt**] If $\theta : R \to S$ is a surjective ring homomorphism, then $\theta(Z(R)) \subseteq Z(S)$.

Solution: It is true: Let $r \in Z(R)$. We show that $\theta(r)$ commutes with every element $s \in S$. Let $s \in S$. Since θ is surjective, there exists an $r_1 \in R$ such that $\theta(r_1) = s$. Therefore we have

$$\theta(r)s = \theta(r)\theta(r_1) = \theta(rr_1) = \theta(r_1 r) = \theta(r_1)\theta(r) = s\theta(r).$$

(d) **[1 pt]** The polynomial $x^{100} + 10x^{50} - 5x^{25} + 15$ is irreducible in $\mathbb{R}[x]$.

Solution: It is false. Every non-constant polynomial in $\mathbb{R}[x]$ factors as a product of polynomials of degree at most two. Therefore the above polynomial should be reducible in $\mathbb{R}[x]$.

(e) **[1 pt]** There exists a polynomial $f(x) \in \mathbb{Q}[x]$ such that $\deg(f(x)) = 2011$ and $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Solution: It is true. One can consider the polynomial $x^{2011} + 3$. From the Eisenstein criterion for $p = 3$, it follows that this polynomial is irreducible in $\mathbb{Q}[x]$.

4. In each of the following, give an appropriate example. You do not need to justify your example.

(a) [**1 pt**] Give an example of a nonzero nilpotent element in the ring $\mathrm{M}_2(\mathbb{R})$. (Recall that $\mathrm{M}_2(\mathbb{R})$ is the ring of 2×2 matrices with real entries.)

Solution: For example, consider the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) [**1 pt**] Give an example of a nonzero proper ideal of the ring $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Solution: For example, consider the ideal $\mathbb{Z} \times \mathbb{Z} \times \{0\}$.

(c) [**1 pt**] Give an example of a ring homomorphism $\theta : \mathbb{C} \to \mathbb{C}$ other than the identity map.

Solution: For example, $\theta(z) = \overline{z}$, the complex conjugation.

(d) [**1 pt**] Give an example of a subring of $\mathbb{Z} \times \mathbb{Z}$ other than $\{(0,0)\}$ and $\mathbb{Z} \times \mathbb{Z}$.

Solution: For example, the subset of elements $\{ \ (x, x) \mid x \in \mathbb{Z} \ \}$ is a subring.

(e) [**1 pt**] Give an example of an infinite ring R such that $\mathrm{char}(R) = 3$.

Solution: For example, consider the ring $\mathbb{Z}_3[x]$.

5. [**2 pts**] Let R be a ring. Prove that the map

$$\theta : R[x] \to R$$

defined by

$$\theta(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n) = a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$$

is a ring homomorphism. What is $\ker(\theta)$?

Solution: Clearly, this map is the evaluation map ϕ_a for $a = -1$. More precisely, $\theta(f(x)) = f(-1)$. Since $-1 \in Z(R)$, the Evaluation Theorem implies that this map is a ring homomorphism.

The kernel of this map is the set of polynomials $f(x) \in R[x]$ such that $f(-1) = 0$.

6. **(Bonus Question)**

(a) [**1 pt**] Let R be a commutative ring. Prove that if two elements $r, s \in R$ are nilpotent, then $r + s$ is also nilpotent.

Solution: Suppose that $r^m = 0$ and $s^n = 0$. Then consider $(r+s)^{m+n}$. Since R is commutative, every term in the expansion of this expression is of the form $r^p b^q$ where $p + q = m + n$. It follows that either $p \geq m$ or $q \geq n$, and consequently $r^p b^q = 0$. This means that $(r+s)^{m+n} = 0$.

(b) [**1 pt**] By an example, show that the statement of part (a) is not true if R is not assumed to be commutative.

Solution: For example in $R = \mathrm{M}_2(\mathbb{R})$ we can consider $r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $s = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Clearly $r^2 = s^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $r + s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is invertible and therefore it is not nilpotent.