University of Ottawa

Department of Mathematics and Statistics

MAT 3143: Ring Theory
Professor: Hadi Salmasian

Final Exam

April 18, 2011

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have three hours to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You are strongly recommended to write in **pen**, not pencil.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	11	12	Total
Maximum	4	4	4	5	4	4	4	5	6	4	4	2	50
Grade													

1. [**4 pts**] Factor the polynomial $f(x) = 2x^4 + x + 2$ into a product of irreducible polynomials in $\mathbb{Z}_5[x]$.

2. [**4 pts**] Prove that the polynomial $q(x) = 3x^3 + 5x - 1$ is irreducible in $\mathbb{Q}[x]$.

3. [**4 pts**] Let $\mathbb{F}$ be a field. Find all of the idempotent elements of the ring $R = \mathbb{F}[x]/A$, where A is the ideal generated by x^2, i.e., $A = \langle x^2 \rangle$.

4.

(a) **[2 pts]** Write the statement of the Chinese Remainder Theorem for any ring R.

(b) **[2 pts]** How many units does $\mathbb{Z}_{65}$ have? You should justify your answer.

(c) **[1 pts]** How many ideals does $\mathbb{Z}_{65}$ have? You should justify your answer.

5.

(a) **[2 pts]** Write the definition of a Euclidean domain.

(b) **[2 pts]** Prove the following theorem : every Euclidean domain is a PID.
(Recall that PID means principal ideal domain.)

6.

(a) **[2 pts]** Let $\mathbb{F} \subseteq \mathbb{E}$ be a field extension and $u \in \mathbb{E}$ be algebraic over $\mathbb{F}$. Write the definition of the minimal polynomial of u over $\mathbb{F}$.

(b) **[2 pts]** Prove the following Theorem :

Let $\mathbb{F} \subseteq \mathbb{E}$ be a finite extension and $[\mathbb{E} : \mathbb{F}] = n$. If $u \in \mathbb{E}$ then there exists a nonzero polynomial $f(x) \in \mathbb{F}[x]$ such that $f(u) = 0$ and $\deg(f(x)) \leq n$.

7.

(a) **[2 pts]** Let $u = \sqrt{3+\sqrt{12}}$. Find the minimal polynomial of u over $\mathbb{Q}$. You should justify your answer.

(b) **[2 pts]** Let $v = \sqrt{3-\sqrt{12}}$. Explain why $\mathbb{Q}(u) \simeq \mathbb{Q}(v)$. Is it also true that $\mathbb{Q}(u) = \mathbb{Q}(v)$? Justify your answer.

8. [**5 pts**] Find a splitting field $\mathbb{Q} \subseteq \mathbb{E}$ for $f(x) = (x^2 - 3)(x^2 + x + 1)$, and compute $[\mathbb{E} : \mathbb{Q}]$. You should justify your answers.

9. Let $i = \sqrt{-1}$ and consider the ring

$$\mathbb{Z}[\,i\,] = \{\ a + bi\ :\ a, b \in \mathbb{Z}\ \}$$

of Gaussian integers.

(a) **[4 pts]** Prove that $1 + 2i$ is a prime element of $\mathbb{Z}[\,i\,]$.

(b) **[2 pts]** Let $r \in \mathbb{Z}[\,i\,]$ be a prime element. Prove that $\overline{r}$ is also a prime element of $\mathbb{Z}[\,i\,]$. (Here $\overline{r}$ means the complex conjugate of r.)

10. Let p be a prime number.

(a) **[2 pts]** Are $GF(p^2)$ and $\mathbb{Z}_p \times \mathbb{Z}_p$ isomorphic? You should justify your answer, i.e., either prove or disprove this statement.

(b) **[2 pts]** How many subfields does $GF(p^8)$ have? You should justify your answer.

11. Let $\mathbb{O} = \{\ldots, -3, -1, 1, 3, \ldots\}$ be the set of odd integers and $\mathbb{P} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ be the set of even integers. Consider the ring

$$R = \{ \ \frac{m}{n} \ : \ m \in \mathbb{Z} \text{ and } n \in \mathbb{O} \ \}$$

with the usual addition and multiplication of rational numbers.

(a) [**1 pt**] Find all of the units of R.

(b) [**2 pts**] Prove that the set

$$A = \{ \ \frac{m}{n} \ : \ m \in \mathbb{P} \text{ and } n \in \mathbb{O} \ \}$$

is a maximal ideal of R.

(c) [**1 pt**] Does R have any maximal ideals other than A? Justify your answer.

12. **[2 pts]** Prove that for every positive integer n, the polynomial $x^n - \sqrt{2}$ is irreducible over $\mathbb{Q}(\sqrt{2})$.

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