MAT 3143: Test (Febr 09)

1) (a) State the definitions of
(i) a maximal ideal, and
(ii) a simple ring.

[4] Solutions (i) An ideal M of a ring R is maximal if $M \neq R$ and if for every ideal I of R with $M \subset I \subset R$ either $M = I$ or $I = R$.
(ii) A ring R is simple if $R \neq \{0\}$ and the only ideals of R are $\{0\}$ and R itself

[6] (b) Let I be an ideal of a ring R. Prove that I is maximal in R if and only if R/I is a simple ring.

Solution: Let I be a maximal ideal of R. Then $R/I \neq \{0\}$ since $I \neq R$. Let J be an ideal of R/I. Then $J = A/I$ for a (unique) ideal A of R with $I \subset A \subset R$. By maximality of I, either $A = I$, ie $J = I/I = \{0\}$, or $A = R$, ie $J = R/I$.

Conversely, let R/I be a simple ring. Since $R/I \neq \{0\}$, it follows that $R \neq I$. Let now $J \trianglelefteq R$ with $I \subset J \subset R$. Then J/I is an ideal of R/I. By simplicity of R/I therefore $J/I = \{0\}$, ie $J = I$, or $J/I = R/I$, ie $J = R$.

2) (a) State the definition of a general ring homomorphism and a ring homomorphism.

[3] Solution: Let $\theta: R_1 \to R_2$ be a map from a ring R_1 to a ring R_2. Then θ is called a general ring homomorphism if $\theta(r_1+s_1) = \theta(r_1) + \theta(s_1)$ and $\theta(r_1 s_1) = \theta(r_1)\theta(s_1)$ for all $r_1, s_1 \in R_1$.

A ring homomorphism is a general ring homomorphism which also satisfies $\theta(1_{R_1}) = 1_{R_2}$ where 1_{R_1} and 1_{R_2} are the identity elements of R_1 and R_2 resp.

[4] (b) Show that every surjective general ring homomorphism is a ring homomorphism.

Solution: Let $\theta: R \to S$ be a surjective general ring homomorphism. We need to show that $\theta(1_R) = 1_S$ where 1_R and 1_S are the identity elements of R and S respectively. Let $s \in S$. Then $s = \theta(r)$ for some $r \in R$, whence $\theta(1_R)s = \theta(1_R)\theta(r) = \theta(1_R r) = \theta(r) = s$ and, similarly, $s\theta(1_R) = \theta(r)\theta(1_R) = \theta(1_R r) = \theta(r) = s$. Since the identity element in S is unique, these two equations imply that $\theta(1_R) = 1_S$.

[5] (c) Determine all ring homomorphisms $\mathbb{Z}_8 \to \mathbb{Z}_4$

Solution: Let $[m]_8$ resp. $[m]_4$ the residue classes of m modulo 8 resp. modulo 4. Since $[1]_8$ generates $\mathbb{Z}_8$ as abelian group and since a ring homomorphism necessarily maps $[1]_8$ to $[1]_4$, we necessarily have $\theta([m]_8) = \theta(m[1]_8) = m\,\theta([1]_8) = m[1]_4 = [m]_4$, ie $\theta([m]_8) = [m]_4$ for all $m \in \mathbb{Z}_8$.

This map is well-defined: $[m]_8 = [n]_8 \Leftrightarrow 8 \mid m-n \Rightarrow 4 \mid m-n \Leftrightarrow [m]_4 = [n]_4$. Also, it is a ring homomorphism: $\theta([m]_8 \pm [n]_8) = \theta([m \pm n]_8) = [m \pm n]_4 = [m]_4 \pm [n]_4 = \theta([m]_8) \pm \theta([n]_8)$, and $\theta([1]_8) = [1]_4$. It is onto since $[m]_4 = \theta([m]_8)$ for every $[m]_4 \in \mathbb{Z}_4$. In sum, the only ring homomorphism $\theta: \mathbb{Z}_8 \to \mathbb{Z}_4$ is the map $\theta([m]_8) = [m]_4$.

3) State the theorem on the existence and properties of the greatest common divisor of two polynomials $f, g \in F[x]$ and prove it.

Solution (a) Theorem: Let $f, g \in F[x]$ be non-zero polynomials. Then there exists a unique polynomial $d \in F[x]$ satisfying

(i) d is monic

(ii) $d \mid f$ and $d \mid g$,

(iii) if $h \mid f$ and $h \mid g$, then $h \mid d$,

(iv) $d = uf + vg$ for some $u, v \in F[x]$.

(b) Proof The set $X = \{uf + vg : u, v \in F[x]\}$ is $\neq \emptyset$, e.g. $f = 1f + 0g \in X$. It follows that X contains a monic polynomial: if $p \in X$ with leading coefficient a, then $\frac{1}{a} p \in X$ is monic. Let $d \in X$ be a monic polynomial in X of minimal degree. Then (i) and (iv) holds.

Proof of (iii): Suppose $hp = f$ and $hq = g$. From $d = uf + vg$ we get $d = uhp + vhq = (up + vq)h$, whence $h \mid d$

Proof of (iv): We apply division with remainder to f, d, ie we write $f = qd + r$ where $r = 0$ or $\deg(r) < \deg(d)$. Then $r = f - qd = f - q(uf + vg) = (1 - qu)f + (-qv)g \in X$. Suppose $r \neq 0$ and let b be its leading coefficient. Then $b^{-1} r \in X$ is monic. But $\deg(b^{-1}r) = \deg(r) < \deg(d)$, contradiction. Hence $r = 0$, proving $d \mid f$. By symmetry $d \mid g$.

Uniqueness: If d and d_1 have the properties (i) - (iii), then $d \mid d_1$ and $d_1 \mid d$ by (ii) & (iii) for d and d_1. Since d and d_1 are monic, we have $d = d_1$.

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4) Give an example of [no justification required]

(a) a subring of $M_2(\mathbb{C})$, the ring of 2x2 complex matrices.

<u>Solution</u>: $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$

(b) a ring which is not an integral domain:

<u>Solution</u>: $\mathbb{Z}_6$ (in $\mathbb{Z}_6$ we have $\overline{3} \cdot \overline{2} = \overline{0}$ but $\overline{3} \neq \overline{0}$ and $\overline{2} \neq \overline{0}$)

(c) a prime ideal which is not maximal:

<u>Solution</u> $R = \mathbb{Z}$, $\{0\}$ is a prime ideal since $\mathbb{Z}/\{0\} = \mathbb{Z}$ is an integral domain, but $\{0\}$ is not a maximal ideal since, for example, $\{0\} \subsetneq 2\mathbb{Z} \subsetneq \mathbb{Z}$.

(d) an irreducible polynomial in $\mathbb{F}[x]$ where $\mathbb{F}$ is an arbitrary field.

<u>Solution</u> $x \in F[x]$.

(e) a family of irreducible polynomials in $\mathbb{Q}[x]$ which has unbounded degree

<u>Solution</u> For any prime p the p^{th} cyclotomic polynomial $\Phi_p(x) = x^{p-1} + x^{p-2} + \dots + x + 1$ is irreducible. Thus $\{\Phi_p(x) : p \text{ prime}\}$ is a family of irreducible polynomials of unbounded degree.