University of Ottawa

Department of Mathematics and Statistics

MAT 3143: Ring Theory

Professor Erhard Neher

Final Exam (April 2009)

Family Name: ______________________________

First Name: ______________________________

Student number: ______________________________

Instructions:

- The exam has 12 pages and 6 questions. If you do not have enough space, use the back of the pages. Every problem has two pages, except the last one.
- You have 180 minutes. The exam is out of 64 points.
- No books, notes or calculators are allowed.

Question	1	2	3	4	5	6	Total
maximal	10	10	10	11	13	10	64
your score							

1. (10 points) (a) Find polynomials q, $r \in \mathbb{Z}_6[x]$ with $r = 0$ or $\deg(r) < 2$ such that $4x^4 + 3x^3 + x + 1 = q(x^2 + 3x + 1) + r$.

(b) For which primes p is $x - 1$ a factor of $3x^5 + 5x^3 + 4x + 2$ in $\mathbb{Z}_p[x]$?

(c) Find all irreducible $f \in \mathbb{Z}_2[x]$ with $\deg(f) = 3$.

(d) Show that $7x^3 + 3x^2 + 2x + 5$ is irreducible in $\mathbb{Q}[x]$.

2. (10 points) (a) Determine the units in $\mathbb{Z}(\sqrt{-3})$ (Hint: Use $N(m+n\sqrt{-3}) = m^2+3n^2$ for $m, n \in \mathbb{Z}$.)

(b) Show that $1 + \sqrt{-3}$ is irreducible in $\mathbb{Z}(\sqrt{-3})$, but not prime.

3. (10 points) Let R be a PID. Prove that the following are equivalent for $0 \neq p \in R \setminus R^{\times}$:

(i) p is a prime element,

(ii) $R/\langle p\rangle$ is a field,

(iii) $R/\langle p\rangle$ is an integral domain.

Here $\langle p\rangle = \{ap : a \in R\}$ is the ideal generated by p.

4. (11 points) (a) Let E/F be a field extension. Define the notions of an algebraic element and its element minimal polynomial.

(b) Show that $(1 - i)^{1/3}$ is algebraic over $\mathbb{Q}$ and determine its minimal polynomial.

(c) Let E/F be an algebraic extension, and let $R \subset E$ be a subring with $F \subset R$. Show that R is a field.

5. (13 points) (a) Let F be a field. Give the definition of a splitting field of a polynomial $f \in F[x]$ of degree ≥ 1.

(b) Find a splitting field E of $f = (x^2 - 2)(x^2 + 1) \in \mathbb{Q}[x]$, determine $[E : \mathbb{Q}]$ and a basis of the $\mathbb{Q}$-vector space $E/\mathbb{Q}$.

(c) Let F be a field with $|F| = p^n$, p a prime. Prove that F is a splitting field of $f = x^{p^n} - x$ over $\mathbb{Z}_p$.

6. (10 points) Give an example of the following (proofs are not required)

(a) a primitive element of $\mathbb{Z}_7$:

(b) an algebraically closed field different from $\mathbb{C}$:

(c) an algebraic extension of infinite degree:

(d) a transcendental element:

(e) a UFD which is not a PID:

(f) an Euclidean domain:

(g) an extension of degree 5:

(h) an irreducible $f \in \mathbb{R}[x]$, $\deg(f) \geq 1.8$:

(i) a maximal ideal in $\mathbb{C}[x]$:

(j) a division ring which is not a field: