Extra office hours: Thurs 2-4pm

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What we called "elementary divisors" should really be called "invariant factors".

$\boxed{Q}$ What is the connection between the invariant factors (what we called elementary divisors) mentioned after Thm. 7.25 and those that appear in the Smith normal form?

$\boxed{A}$ The invariant factors (elementary divisors) of $T \in \mathrm{End}_K V$ in the sense of Thm 7.25 are those of the matrix $xI - A$, where A is a matrix of T in some basis, in the sense of Smith normal form.

- One can use this to compute the rational canonical form
- to get the basis in which an operator is in rational canonical form see Section 8.6

Example: Find a Jordan canonical form of

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

$$\det(xI - A) = \begin{vmatrix} x-2 & 1 & 1 \\ -1 & x-2 & 0 \\ 1 & 0 & x-2 \end{vmatrix} = x^3 - 6x^2 + 12x - 8$$

2 is a root of

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$$\begin{array}{r} x^2 - 4x + 4 \\ x-2 \;\overline{)\; x^3 - 6x^2 + 12x - 8} \\ \underline{x^3 - 2x^2 \qquad\qquad} \\ -4x^2 + 12x - 8 \\ \underline{-4x^2 + 8x \quad} \\ 4x - 8 \end{array}$$

$\therefore$ $x^3 - 6x^2 + 12x - 8$
$= (x-2)(x^2 - 4x + 4)$
$= (x-2)^3$

So only eigenvalue is $\lambda = 2$, with mult 3.

Let's find dim E_2.

$$2I - A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim E_2 = 1$$

$\therefore$ Jordan canonical form is

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Possibilities for Jordan canonical form:

Size of matrix	Distinct eigenvalues w/ mult	JCF		
1×1	λ_1 (mult 1)	(λ_1)		
2×2	λ_1, λ_2 (both mult 1)	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$		
	λ_1 (x2)	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$ $\dim E_{\lambda_1} = 2$	$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$ $\dim E_{\lambda_1} = 1$	
3×3	$\lambda_1, \lambda_2, \lambda_3$	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$		
	λ_1 (x2), λ_2	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ $\dim E_{\lambda_1} = 2$	$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ $\dim E_{\lambda_1} = 1$	
	λ_1 (x3)	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$ $\dim E_{\lambda_1} = 3$	$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$ $\dim E_{\lambda_1} = 2$	$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$ $\dim E_{\lambda} = 1$

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Size of matrix (A)	Distinct eigenvalues	JCF	
4×4	λ_1 $(\times 4)$ (other possibilities reduce to cases above)	$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$ $\dim E_{\lambda_1} = 4$	$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$ $\dim E_{\lambda_1} = 3$
		$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$ $\dim E_{\lambda_1} = 2$, $\dim \operatorname{Ker}(\lambda_1 I - A)^2 = 3$	$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$ $\dim E_{\lambda_1} = 2$, $\dim \operatorname{Ker}(\lambda_1 I - A)^2 = 4$
		$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$ $\dim E_{\lambda_1} = 1$	

Q What is the rational canonical form of

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

A We found JCF $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. So $\min_A(x) = (x-2)^3$

$\therefore$ $\min_A(x)$ is only invariant factor (elem. divisor)

(Recall $d_1(x) \mid d_2(x) \mid \cdots \mid d_s(x) = \min_A(x)$ and $\sum \deg d_i =$ size of matrix)

$(x-2)^3 = x^3 - 6x^2 + 12x - 8$

$\therefore$ rational canon. form is $\cancel{\begin{pmatrix} 0 & & & 8 \\ 1 & 0 & & -12 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & \end{pmatrix}}$ $\begin{pmatrix} 0 & 0 & 8 \\ 1 & 0 & -12 \\ 0 & 1 & 6 \end{pmatrix}$

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Other possible Ch 7/8 questions:

- use the structure theorem (abstract statement) to deduce something about a module of finite type over a ED
- find a basis of a col module (or of a span of some elements of a module)
- find a basis of a solution set to some equations with coeffs in a ED
- find a Smith normal form and the invariant factors (elementary divisors) of some matrix

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$\boxed{Q}$ Exercise 9.4.6(i). U, V f.d. v.s. $f \in U^*$, $g \in V^*$.

Show $\exists!$ linear form $f \otimes g : U \otimes_K V \to K$ s.t.

$$(f \otimes g)(u \otimes v) = f(u) g(v) \in U \times V.$$

$\boxed{A}$ Define map $U \times V \to K$ by $(u, v) \mapsto f(u) g(v)$.

Exercise: Show this map is bilinear.

Then, by universal prop. of $\otimes$, we have unique map

$$U \otimes_K V \to K \text{ s.t. } u \otimes v \mapsto f(u) g(v).$$

This is our $f \otimes g$.

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Q (ii) Show $U^* \otimes_K V^* \cong (U \otimes_K V)^*$.

A Exercise: Show $B : U^* \times V^* \to (U \otimes_K V)^*$, $(f,g) \mapsto f \otimes g$ is bilinear. So it induces map $U^* \otimes_K V^* \mapsto (U \otimes_K V)^*$

Now show its image generates $(U \otimes_K V)^*$ as a v.s.

Let $\{u_1, \ldots, u_m\}$, $\{v_1, \ldots, v_n\}$ be bases of U and V.

$\therefore$ $\{u_i \otimes v_j \,;\, 1 \le i \le m,\ 1 \le j \le n\}$ is a basis of $U \otimes_K V$

Let $\underset{B=}{}\{f_1, \ldots, f_m\}$, $\{g_1, \ldots, g_n\}$ be dual bases of U^* and V^*

Then $(f_a \otimes g_b)(u_i \otimes v_j) = f_a(u_i) \otimes g_b(v_j) = \delta_{ai}\delta_{bj}$

$\therefore$ $\{f_a \otimes g_b \,;\, 1 \le a \le m,\ 1 \le b \le n\}$ is basis of $(U \otimes_K V)^*$ dual to B, hence it generates $(U \otimes_K V)^*$. Since each $f_a \otimes g_b$ is in the image of B, we're done.

Finally,

$$\begin{aligned}
\dim (U^* \otimes_K V^*) &= (\dim U^*)(\dim V^*) \\
&= (\dim U)(\dim V) \\
&= \dim (U \otimes_K V) \\
&= \dim (U \otimes_K V)^*
\end{aligned}$$

$\therefore$ the map $U^* \otimes_K V^* \to (U \otimes_K V)^*$ induced by B is a surjective linear map between v.s of the same dim. So it is an isomorphism.

Q Exercise 10.1.4. V inn. prod space w/ inn. prod $\langle , \rangle$.

(i) Verify that, for $u \in V$, the func. $f_u : V \to \mathbb{R}$ given by

$$f_u(v) = \langle u, v \rangle \quad \forall \ v \in V.$$

is in V^* and that the map $\varphi : V \to V^*$ given by $\varphi(u) = f_u \ \forall \ u \in V$ is an isom of $\mathbb{R}$-v.s.

A Easy to check that $f_u \in V^*$ (i.e. is linear).

Easy to check that φ is linear.

Since $\dim V = \dim V^*$, suffices to show φ is injective.

Suppose $\varphi(u) = 0$ for some $u \in V$. So $f_u = 0$

$$\therefore \quad \langle u, v \rangle = f_u(v) = 0 \ \forall \ v \in V$$

$\Rightarrow u = 0.$

Q (ii). $B = \{u_1, \ldots, u_n\}$ orthonormal basis of V. Show that basis of V^* dual to B is $B^* = \{f_1, \ldots, f_n\}$, where

$$f_i := \varphi(u_i) = f_{u_i} \text{ for } i = 1, \ldots, n.$$

A We have $f_i(u_j) = f_{u_i}(u_j) = \langle u_i, u_j \rangle = \delta_{ij}$. $\square$