MAT 3141 – Fall 2012

Midterm Test

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For all questions, you must justify your answers in order to receive full marks.

Question 1. **(6 points)** Let D be the restriction to $V = \langle e^x, xe^x, x^2e^x\rangle_{\mathbb{R}}$ of the operator of differentiation in $\mathbb{C}_\infty(\mathbb{R})$.

(a) Show that $\mathcal{B} = \{e^x, xe^x, x^2e^x\}$ is a basis of V and determine $[\,D\,]_{\mathcal{B}}$.

(b) Let $p(x) = x^2 - 1$ and $q(x) = x - 1$. Compute $[p(D)]_{\mathcal{B}}$ and $[q(D)]_{\mathcal{B}}$. Do these matrices commute? Why or why not?

Question 2. **(4 points)** Suppose A is a commutative ring and that M and N are A-modules.

(a) Show that if there exists a surjective homomorphism of A-modules from M to N, then $\mathrm{Ann}(M) \subseteq \mathrm{Ann}(N)$.

(b) Show that if there exists an injective homomorphism of A-modules from M to N, then $\mathrm{Ann}(N) \subseteq \mathrm{Ann}(M)$.

Question 3. **(5 points)** Suppose V is a vector space of dimension 6 over $\mathbb{Q}$ and let $T : V \to V$ be a linear operator. Suppose that, for the corresponding structure of a $\mathbb{Q}[x]$-module on V, we have

$$V = \mathbb{Q}[x]\mathbf{v}_1 \oplus \mathbb{Q}[x]\mathbf{v}_2,$$

with $\mathrm{Ann}(\mathbf{v}_1) = (x^4 - x^3 + x^2)$ and $\mathrm{Ann}(\mathbf{v}_2) = (x^2 - 1)$.

(a) Give a nonzero polynomial $p(x) \in \mathrm{Ann}(V)$.

(b) Find a basis $\mathcal{B}$ of V such that the matrix $[T]_{\mathcal{B}}$ is block diagonal with the companion matrices of powers of monic irreducible polynomials on the diagonal (and give the matrix $[T]_{\mathcal{B}}$).

Question 4. **(2 points)** Suppose A is a euclidean domain. An ideal I of A is called *prime* if, for all $a, b \in A$,

$$ab \in I \implies (a \in I \text{ or } b \in I).$$

Show that if q is an irreducible element of A, then the ideal (q) generated by q is prime.

Question 5. **(3 points)** Let V be a vector space over a field K. Let $T \in \mathrm{End}_K(V)$ and consider the corresponding structure of a $K[x]$-module on V. Suppose S is another element of $\mathrm{End}_K(V)$ such that $S \circ T = T \circ S$. Show that S is a homomorphism of $K[x]$-modules from V to itself. *Note:* The $K[x]$-module structure on V is determined by T, not S.