MAT 3141 – Fall 2012

Midterm Test – Solutions

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October 30, 2012

For all questions, you must justify your answers in order to receive full marks.

QUESTION 1. **(6 points)** Let D be the restriction to $V = \langle e^x, xe^x, x^2e^x \rangle_{\mathbb{R}}$ of the operator of differentiation in $\mathbb{C}_\infty(\mathbb{R})$.

(a) Show that $\mathcal{B} = \{e^x, xe^x, x^2e^x\}$ is a basis of V and determine $[\,D\,]_{\mathcal{B}}$.

(b) Let $p(x) = x^2 - 1$ and $q(x) = x - 1$. Compute $[p(D)]_{\mathcal{B}}$ and $[q(D)]_{\mathcal{B}}$. Do these matrices commute? Why or why not?

Solution:

(a) By definition, $\mathcal{B}$ spans V. Thus, to show that $\mathcal{B}$ is a basis of V, it suffices to show that $\mathcal{B}$ is linearly independent. Suppose $a, b, c \in \mathbb{R}$ such that

$$ae^x + bxe^x + cx^2e^x = 0.$$

Then $e^x(a + bx + cx^2) = 0$. Since $e^x \neq 0$ for all $x \in \mathbb{R}$, we have $a + bx + cx^2 = 0$ for all $x \in \mathbb{R}$. This implies $a = b = c = 0$ (since any nonzero polynomial of degree ≤ 3 has at most three real roots). Thus $\mathcal{B}$ is a linearly independent set.

We have

$$D(e^x) = e^x, \quad D(xe^x) = e^x + xe^x, \quad D(x^2e^x) = 2xe^x + x^2e^x,$$

and so

$$[D]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) We have

$$[p(D)]_{\mathcal{B}} = p([D]_{\mathcal{B}}) = [D]_{\mathcal{B}}^2 - I = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} - I = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[q(D)]_{\mathcal{B}} = p([D]_{\mathcal{B}}) = [D]_{\mathcal{B}} - I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $[p(D)]_{\mathcal{B}} = p([D]_{\mathcal{B}})$ and $[q(D)]_{\mathcal{B}} = q([D]_{\mathcal{B}})$, we see that $[p(D)]_{\mathcal{B}}$ and $[q(D)]_{\mathcal{B}}$ are both polynomials in the matrix $[D]_{\mathcal{B}}$. Since any matrix commutes with itself, this implies that $[p(D)]_{\mathcal{B}}$ and $[q(D)]_{\mathcal{B}}$ commute. Another way to see that they commute is to note that

$$[p(D)]_{\mathcal{B}}[q(D)]_{\mathcal{B}} = [p(D)q(D)]_{\mathcal{B}} = [(pq)(D)]_{\mathcal{B}} = [(qp)(D)]_{\mathcal{B}} = [q(D)p(D)]_{\mathcal{B}} = [q(D)]_{\mathcal{B}}[p(D)]_{\mathcal{B}}.$$

Alternatively, one can directly compute the product of the two matrices in both orders and see that they are the same.

QUESTION 2. **(4 points)** Suppose A is a commutative ring and that M and N are A-modules.

(a) Show that if there exists a surjective homomorphism of A-modules from M to N, then $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(N)$.

(b) Show that if there exists an injective homomorphism of A-modules from M to N, then $\operatorname{Ann}(N) \subseteq \operatorname{Ann}(M)$.

Solution:

(a) Let $\varphi : M \to N$ be a surjective homomorphism of A-modules and let $a \in \operatorname{Ann}(M)$. Choose $\mathbf{v} \in N$. Since φ is surjective, there exists $\mathbf{u} \in M$ such that $\varphi(\mathbf{u}) = \mathbf{v}$. Then $a\mathbf{v} = a\varphi(\mathbf{u}) = \varphi(a\mathbf{u}) = \varphi(0) = 0$. Since our choice of $\mathbf{v} \in N$ was arbitrary, this proves that $a \in \operatorname{Ann}(N)$. Thus $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(N)$.

(b) Let $\varphi : M \to N$ be an injective homomorphism of A-modules and let $a \in \operatorname{Ann}(N)$. Choose $\mathbf{u} \in M$. Then $\varphi(a\mathbf{u}) = a\varphi(\mathbf{u}) = 0$, since $\varphi(\mathbf{u}) \in N$. Because φ is injective, this implies that $a\mathbf{u} = 0$. Since our choice of $\mathbf{u} \in M$ was arbitrary, this shows that $a \in \operatorname{Ann}(M)$. Hence $\operatorname{Ann}(N) \subseteq \operatorname{Ann}(M)$.

QUESTION 3. **(5 points)** Suppose V is a vector space of dimension 6 over $\mathbb{Q}$ and let $T : V \to V$ be a linear operator. Suppose that, for the corresponding structure of a $\mathbb{Q}[x]$-module on V, we have

$$V = \mathbb{Q}[x]\mathbf{v}_1 \oplus \mathbb{Q}[x]\mathbf{v}_2,$$

with $\operatorname{Ann}(\mathbf{v}_1) = (x^4 - x^3 + x^2)$ and $\operatorname{Ann}(\mathbf{v}_2) = (x^2 - 1)$.

(a) Give a nonzero polynomial $p(x) \in \operatorname{Ann}(V)$.

(b) Find a basis $\mathcal{B}$ of V such that the matrix $[T]_\mathcal{B}$ is block diagonal with the companion matrices of powers of monic irreducible polynomials on the diagonal (and give the matrix $[T]_\mathcal{B}$).

Solution:

(a) Since $V = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathbb{Q}[x]}$, we have $\operatorname{Ann}(V) = \operatorname{Ann}(\mathbf{v}_1) \cap \operatorname{Ann}(\mathbf{v}_2)$. Thus $p(x) = (x^4 - x^3 + x^2)(x^2 - 1) \in \operatorname{Ann}(V)$.

(b) We have $x^4 - x^3 + x^2 = x^2(x^2 - x + 1)$, where x and $x^2 - x + 1$ are irreducible in $\mathbb{Q}[x]$ (the polynomial $x^2 - x - 1$ is irreducible because its discriminant is negative). Thus,

$$\mathbb{Q}[x]\mathbf{v}_1 = \mathbb{Q}[x]\mathbf{v}_1' \oplus \mathbb{Q}[x]\mathbf{v}_1'',$$

where $\mathbf{v}_1' = (x^2 - x + 1)\mathbf{v}_1 = T^2(\mathbf{v}_1) - T(\mathbf{v}_1) + \mathbf{v}_1$ and $\mathbf{v}_1'' = x^2(\mathbf{v}_1) = T^2(\mathbf{v}_1)$ satisfy

$$\operatorname{Ann}(\mathbf{v}_1') = (x^2) \quad \text{and} \quad \operatorname{Ann}(\mathbf{v}_1'') = (x^2 - x + 1).$$

Similarly, $x^2 - 1 = (x + 1)(x - 1)$, where $x + 1$ and $x - 1$ are irreducible. Thus,

$$\mathbb{Q}[x]\mathbf{v}_2 = \mathbb{Q}[x]\mathbf{v}_2' \oplus \mathbb{Q}[x]\mathbf{v}_2'',$$

where $\mathbf{v}_2' = (x - 1)\mathbf{v}_2 = T(\mathbf{v}_2) - \mathbf{v}_2$ and $\mathbf{v}_2'' = (x + 1)\mathbf{v}_2 = T(\mathbf{v}_2) + \mathbf{v}_2$ satisfy

$$\operatorname{Ann}(\mathbf{v}_2') = (x + 1) \quad \text{and} \quad \operatorname{Ann}(\mathbf{v}_2'') = (x - 1).$$

We therefore have

$$V = \mathbb{Q}[x]\mathbf{v}_1' \oplus \mathbb{Q}[x]\mathbf{v}_1'' \oplus \mathbb{Q}[x]\mathbf{v}_2' \oplus \mathbb{Q}[x]\mathbf{v}_2''$$

and $\mathcal{B} = \{\mathbf{v}_1', T(\mathbf{v}_1'), \mathbf{v}_1'', T(\mathbf{v}_1''), \mathbf{v}_2', \mathbf{v}_2''\}$ is a basis of V for which

$$[T]_\mathcal{B} = \begin{pmatrix} 0 & 0 & & & & \\ 1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 1 & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix},$$

with the companion matrices of x^2, $x^2 - x + 1$, $x - 1$ and $x + 1$ on the diagonal.

QUESTION 4. **(2 points)** Suppose A is a euclidean domain. An ideal I of A is called *prime* if, for all $a, b \in A$,

$$ab \in I \implies (a \in I \text{ or } b \in I).$$

Show that if q is an irreducible element of A, then the ideal (q) generated by q is prime.

Solution: Let q be an irreducible element of A and $a, b \in A$ with $ab \in (q)$. Then $q \mid ab$. Since q is irreducible and A is a euclidean domain, this implies that $q \mid a$ or $q \mid b$. Hence $a \in (q)$ or $b \in (q)$.

QUESTION 5. **(3 points)** Let V be a vector space over a field K. Let $T \in \mathrm{End}_K(V)$ and consider the corresponding structure of a $K[x]$-module on V. Suppose S is another element of $\mathrm{End}_K(V)$ such that $S \circ T = T \circ S$. Show that S is a homomorphism of $K[x]$-modules from V to itself. *Note:* The $K[x]$-module structure on V is determined by T, not S.

Solution: Since S is a linear map, we have $S(\mathbf{v}_1 + \mathbf{v}_2) = S(\mathbf{v}_1) + S(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$. Now, for any positive integer n, we have

$$T^n \circ S = T^{n-1} \circ S \circ T = T^{n-2} \circ S \circ T^2 = \cdots = S \circ T^n.$$

So S commutes with all powers of T. Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m \in K[x]$. Then, for $\mathbf{v} \in V$,

$$\begin{aligned} S(p(x)\mathbf{v}) = S(p(T)(\mathbf{v})) = S(a_0\mathbf{v} + a_1 T(\mathbf{v}) + \cdots + a_n T^n(\mathbf{v})) &= S(a_0\mathbf{v}) + S(a_1 T(\mathbf{v})) + \cdots + S(a_n T^n(\mathbf{v})) \\ &= a_0 S(\mathbf{v}) + a_1 T(S(\mathbf{v})) + \cdots + a_n T^n(S(\mathbf{v})) = p(T)(S(\mathbf{v})) = p(x)S(\mathbf{v}). \end{aligned}$$

Thus S is a homomorphism of $K[x]$-modules.