University of Ottawa

Department of Mathematics and Statistics

MAT 3141: Linear Algebra II

Professor: Alistair Savage

Final Exam

December 2012

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have 3 hours to complete this exam.

(b) This exam consists of two parts. Questions 1 through 5 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 6 through 13 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(d) Write your student number at the top of each page in the space provided.

(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(f) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1–5	6	7	8	9	10	11	12	13	Total
Maximum	8	4	3	6	4	4	5	5	5	44
Grade										

Part A: Answer Only Questions

For Questions 1–5, only your final answer will be considered for marks. Write your final answer(s) in the space(s) provided.

1. **(1 pt)** Consider $\mathbb{C}^3$ and $\mathbb{C}^4$ as *real* vector spaces (i.e. over the field $\mathbb{R}$.). What is $\dim_{\mathbb{R}}(\mathbb{C}^3 \otimes_{\mathbb{R}} \mathbb{C}^4)$?

Answer:________________________

2. **(1 pt)** Let V be a vector space over a field K and let $T \in \mathrm{End}_K(V)$. Consider the following statement:

V is a finite dimensional vector space.

Which of the following is true?

(a) The above statement is true whenever V is finitely generated as a $K[x]$-module.
(b) The above statement is false whenever V is finitely generated as a $K[x]$-module.
(c) The above statement is true for some V which are finitely generated as a $K[x]$-module and false for some V which are finitely generated as a $K[x]$-module.

Answer:________________________

3. **(2 pts)** Give a basis (over $\mathbb{Z}$) of the column space of

$$\begin{pmatrix} 4 & -4 & 2 \\ 5 & -5 & 4 \\ -1 & 1 & 0 \\ -4 & 4 & -2 \end{pmatrix}.$$

Answer:________________________

4. **(2 pts)** Find the invariant factors of the matrix

$$\begin{pmatrix} x^2 & 0 & x \\ x^2 - x & x^2 - x & 0 \\ x^3 + x^2 - x & x^2 - x & x^2 \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{Q}[x]).$$

Answer:________________________

5. **(2 pts)** Which of the following statements is/are true for every real vector space V and $T \in \mathrm{End}_{\mathbb{R}}(V)$?

(a) There is a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is in Jordan canonical form.
(b) There is a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is block diagonal with blocks of size 1×1 or 2×2.
(c) If T is orthogonal, then there is an orthonormal basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is a diagonal matrix.
(d) If T is a symmetric, then there is an orthonormal basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

Answer:________________________

Part B: Long Answer Questions

For Questions 6–13, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

6. **(4 pts)** Let $p(x) = x^3 - 3x^2 + 4$ and $q(x) = x^3 - 4x^2 + 5x - 2$, considered as elements of $\mathbb{Q}[x]$. Find $\gcd(p(x), q(x))$ and write it as a $\mathbb{Q}[x]$-linear combination of $p(x)$ and $q(x)$.

7. **(3 pts)** Suppose that M is a module of finite type over a euclidean domain A and that the zero element of M is the only torsion element of M. Show that M is a free A-module. *Note:* Recall that an element $\mathbf{u} \in M$ is *torsion* if there exists a nonzero element $a \in A$ such that $a\mathbf{u} = \mathbf{0}$.

8. **(6 pts)**

(a) Let $T \in \operatorname{End}_{\mathbb{C}} \mathbb{C}^3$ be the linear operator whose matrix in the standard basis of $\mathbb{C}^3$ is

$$A = \begin{pmatrix} -3 & -1 & 1 \\ -1 & -3 & 1 \\ -2 & -2 & 0 \end{pmatrix} \in \operatorname{Mat}_{3\times 3}(\mathbb{C}).$$

Find a Jordan canonical form of T. *Hint:* -2 is an eigenvalue of A.

(b) What is the minimal polynomial of T?

Additional space for Question 8.

9. **(4 pts)** Let A be an arbitrary integral domain and let I be an ideal of A considered as an A-submodule of $A^1 = A$. Show that I is a free A-module if and only I is principal.

10. **(4 pts)** Find a basis for

$$N = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{Z}^3 \ ; \ 3x + 6y - 8z = 0 \right\}$$

as a $\mathbb{Z}$-submodule of $\mathbb{Z}^3$.

11. **(5 pts)** Suppose that U and V are finite-dimensional vector spaces over a field K and that $S \in \mathrm{End}_K(U)$ and $T \in \mathrm{End}_K(V)$.

(a) Show that if $\mathbf{u} \in U$ and $\mathbf{v} \in V$ are both nonzero, then $\mathbf{u} \otimes \mathbf{v}$ is a nonzero element of $U \otimes V$.

(b) Show that if μ and λ are eigenvalues of S and T respectively, then $\mu\lambda$ is an eigenvalue of $S \otimes T$.

(c) If S and T are diagonalizable, is $S \otimes T$ necessarily diagonalizable? Remember to justify your answer (i.e. if it is, prove it; if not, give a counterexample).

Additional space for Question 11.

12. **(5 pts)** Suppose that V is a finite-dimensional inner product space and that

$$T : V \to V$$

is a linear operator. Show that $\ker(T^*) = (\mathrm{Im}(T))^\perp$ and that $\mathrm{Im}(T^*) = (\ker(T))^\perp$.

13. **(5 pts)** Consider the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$ given by

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = u_1 v_1 - \frac{1}{2} u_1 v_2 - \frac{1}{2} u_2 v_1 + u_2 v_2.$$

Let $T \in \operatorname{End}_{\mathbb{R}} \mathbb{R}^2$ be the linear operator whose matrix in the standard basis of $\mathbb{R}^2$ is

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

(a) Show that $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix} \right\}$ is an orthonormal basis of $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$.

(b) Considered as an operator on $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$, is T orthogonal?

(c) Considered as an operator on $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$, is T self-adjoint?

Additional space for Question 13.

This page is intentionally left blank. You may use it as scrap paper.