University of Ottawa

Department of Mathematics and Statistics

MAT 3141: Linear Algebra II
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Final Exam
December 2012

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) This exam consists of two parts. Questions 1 through 5 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 6 through 13 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1–5	6	7	8	9	10	11	12	13	Total
Maximum	8	4	3	6	4	4	5	5	5	44
Grade										

Part A: Answer Only Questions

For Questions 1–5, only your final answer will be considered for marks. Write your final answer(s) in the space(s) provided.

1. **(1 pt)** Consider $\mathbb{C}^3$ and $\mathbb{C}^4$ as *real* vector spaces (i.e. over the field $\mathbb{R}$.). What is $\dim_{\mathbb{R}}(\mathbb{C}^3 \otimes_{\mathbb{R}} \mathbb{C}^4)$?

Answer: 48

2. **(1 pt)** Let V be a vector space over a field K and let $T \in \operatorname{End}_K(V)$. Consider the following statement:

V is a finite dimensional vector space.

Which of the following is true?

(a) The above statement is true whenever V is finitely generated as a $K[x]$-module.
(b) The above statement is false whenever V is finitely generated as a $K[x]$-module.
(c) The above statement is true for some V which are finitely generated as a $K[x]$-module and false for some V which are finitely generated as a $K[x]$-module.

Answer: (c)

3. **(2 pts)** Give a basis (over $\mathbb{Z}$) of the column space of

$$\begin{pmatrix} 4 & -4 & 2 \\ 5 & -5 & 4 \\ -1 & 1 & 0 \\ -4 & 4 & -2 \end{pmatrix}.$$

Answer: $\left\{ \begin{pmatrix} 2 \\ 4 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}$

4. **(2 pts)** Find the invariant factors of the matrix

$$\begin{pmatrix} x^2 & 0 & x \\ x^2 - x & x^2 - x & 0 \\ x^3 + x^2 - x & x^2 - x & x^2 \end{pmatrix} \in \operatorname{Mat}_{3\times 3}(\mathbb{Q}[x]).$$

Answer: x, $x^2 - x$

5. **(2 pts)** Which of the following statements is/are true for every real vector space V and $T \in \operatorname{End}_{\mathbb{R}}(V)$?

(a) There is a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is in Jordan canonical form.
(b) There is a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is block diagonal with blocks of size 1×1 or 2×2.

(c) If T is orthogonal, then there is an orthonormal basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

(d) If T is a symmetric, then there is an orthonormal basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

Answer: (d)

Part B: Long Answer Questions

For Questions 6–13, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

6. **(4 pts)** Let $p(x) = x^3 - 3x^2 + 4$ and $q(x) = x^3 - 4x^2 + 5x - 2$, considered as elements of $\mathbb{Q}[x]$. Find $\gcd(p(x), q(x))$ and write it as a $\mathbb{Q}[x]$-linear combination of $p(x)$ and $q(x)$.

Solution: We use the euclidean algorithm.

$$\begin{aligned} x^3 - 4x^2 + 5x - 2 &= (x^3 - 3x^2 + 4) + (-x^2 + 5x - 6) \\ x^3 - 3x^2 + 4 &= (-x - 2)(-x^2 + 5x - 6) + (4x - 8) \\ -x^2 + 5x - 6 &= (4x - 8)\left(\frac{-1}{4}(x - 3)\right). \end{aligned}$$

Thus $\gcd(p(x), q(x)) = x - 2$ (multiplying $4x - 8$ by $1/2$ to obtain a monic gcd). We have

$$\begin{aligned} x - 2 &= \frac{1}{4}(x^3 - 3x^2 + 4) - \frac{1}{4}(-x - 2)(-x^2 + 5x - 6) \\ &= \frac{1}{4}(x^3 - 3x^2 + 4) - \frac{1}{4}(-x - 2)\big((x^3 - 4x^2 + 5x - 2) - (x^3 - 3x^2 + 4)\big) \\ &= -\frac{1}{4}(x + 1)(x^3 - 3x^2 + 4) + \frac{1}{4}(x + 2)(x^3 - 4x^2 + 5x - 2). \end{aligned}$$

7. **(3 pts)** Suppose that M is a module of finite type over a euclidean domain A and that the zero element of M is the only torsion element of M. Show that M is a free A-module. *Note:* Recall that an element $\mathbf{u} \in M$ is *torsion* if there exists a nonzero element $a \in A$ such that $a\mathbf{u} = \mathbf{0}$.

Solution: Since the result is trivially true if M is the zero module, we assume that M is not the zero module. By the structure theorem, we have $M = A\mathbf{u}_1 \oplus \cdots \oplus A\mathbf{u}_s$ for some nonzero $\mathbf{u}_1, \dots, \mathbf{u}_s \in M$. Thus it suffices to show that $A\mathbf{u}_i$ is a free A-module for each $i = 1, \dots, s$. Now, for $i = 1, \dots, s$, we have

$$a\mathbf{u}_i = 0 \text{ for some nonzero } a \in A \iff \mathbf{u}_i \text{ is torsion.}$$

Since M has no nonzero torsion elements, we see that $\{\mathbf{u}_i\}$ is a basis for $A\mathbf{u}_i$ and we are done.

8. **(6 pts)**

(a) Let $T \in \operatorname{End}_{\mathbb{C}} \mathbb{C}^3$ be the linear operator whose matrix in the standard basis of $\mathbb{C}^3$ is

$$A = \begin{pmatrix} -3 & -1 & 1 \\ -1 & -3 & 1 \\ -2 & -2 & 0 \end{pmatrix} \in \operatorname{Mat}_{3\times 3}(\mathbb{C}).$$

Find a Jordan canonical form of T. *Hint:* -2 is an eigenvalue of A.

(b) What is the minimal polynomial of T?

Solution:

(a) We first compute the characteristic polynomial:

$$\operatorname{char}_T(x) = \det(xI - A) = \det \begin{pmatrix} \lambda+3 & 1 & -1 \\ 1 & \lambda+3 & -1 \\ 2 & 2 & \lambda \end{pmatrix} = (x+2)^3.$$

Thus -2 is the only eigenvalue of A. We next find the dimension of the corresponding eigenspace. So we row reduce.

$$-2I - A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $-2I - A$ has rank 1, the eigenspace has dimension 2. So the Jordan canonical form of A has two Jordan blocks, whose sizes add up to three. Therefore, the only possibility (up to permutation of the Jordan blocks) for the Jordan canonical form of A is

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(b) Since the $\min_T(x)$ divides $\operatorname{char}_T(x)$, we have $\min_T(x) = (x+2)^n$ for some $n \geq 3$. We also know that $\min_T(x)$ is the monic polynomial of minimal degree such that $\min_T(T) = 0$. Now, if $\mathcal{B}$ is the basis of $\mathbb{C}^3$ for which the matrix of T is the Jordan canonical form given above, then

$$[T + 2I]_{\mathcal{B}} = [T]_{\mathcal{B}} + 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This matrix is not zero, but its square is zero. Thus $\min_T(x) = (x+2)^2$.

9. **(4 pts)** Let A be an arbitrary integral domain and let I be an ideal of A considered as an A-submodule of $A^1 = A$. Show that I is a free A-module if and only I is principal.

Solution: Suppose I is a free A-module. Then I has a basis $\mathcal{B}$. If $\mathcal{B} = \varnothing$, then I is the zero ideal, which is principal. Therefore, assume $\mathcal{B} \neq \varnothing$. Suppose a_1 and a_2 are distinct elements of A. Since $a_1 a_2 - a_2 a_1 = 0$, the elements a_1, a_2 are linearly dependent. Thus $\mathcal{B}$ cannot have more than one element. Hence I is principal.

Now suppose that I is principal. If $I = \{0\}$, then it is free with basis $\varnothing$. So we assume $I \neq \{0\}$. Then $I = (a) = Aa$ for some nonzero $a \in A$. So the set $\{a\}$ is a generating set for I. Since $a \neq 0$ and A is an integral domain, we have $ba = 0$ only when $b = 0$. Thus the set $\{a\}$ is also linearly independent. Therefore $\{a\}$ is a basis of I and so I is a free A-module.

10. **(4 pts)** Find a basis for

$$N = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{Z}^3 \; ; \; 3x + 6y - 8z = 0 \right\}$$

as a $\mathbb{Z}$-submodule of $\mathbb{Z}^3$.

Solution: We first find a basis for

$$N' = \left\{ \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \in N \right\} = \left\{ \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \in \mathbb{Z}^3 \; ; \; 6y - 8z = 0 \right\}.$$

Now,

$$6y - 8z = 0 \iff 6y = 8z \iff 3y = 4z \iff y = 4w, \; z = 3w \text{ for some } w \in \mathbb{Z}.$$

Therefore $\{(0,4,3)^t\}$ is a basis of N'.

We now find a generator of the ideal

$$I = \{x \in \mathbb{Z} \; ; \; 3x + 6y - 8z = 0 \text{ for some } y, z \in \mathbb{Z}\}.$$

We have

$$\begin{aligned} x \in I &\iff 3x = -6y + 8z \text{ for some } y, z \in Z \\ &\iff 3x \in (-6, 8) = (2) \\ &\iff 2 \,|\, 3x \\ &\iff 2 \,|\, x. \end{aligned}$$

Thus $I = (2)$. Since $3(2) + 6(-1) - 8(0) = 0$, we have $(2, -1, 0)^t \in N$ and so

$$\left\{ \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$$

is a basis of N.

11. **(5 pts)** Suppose that U and V are finite-dimensional vector spaces over a field K and that $S \in \mathrm{End}_K(U)$ and $T \in \mathrm{End}_K(V)$.

(a) Show that if $\mathbf{u} \in U$ and $\mathbf{v} \in V$ are both nonzero, then $\mathbf{u} \otimes \mathbf{v}$ is a nonzero element of $U \otimes V$.

(b) Show that if μ and λ are eigenvalues of S and T respectively, then $\mu\lambda$ is an eigenvalue of $S \otimes T$.

(c) If S and T are diagonalizable, is $S \otimes T$ necessarily diagonalizable? Remember to justify your answer (i.e. if it is, prove it; if not, give a counterexample).

Solution:

(a) Suppose $\mathbf{u} \in U$ and $\mathbf{v} \in V$ are both nonzero. Then we can extend $\{\mathbf{u}\}$ and $\{\mathbf{v}\}$ to bases $\{\mathbf{u}_1 = \mathbf{u}, \dots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1 = \mathbf{v}, \dots, \mathbf{v}_n\}$ of U and V respectively. Then $\{\mathbf{u}_i \otimes \mathbf{u}_j \ ; \ 1 \le i \le m,\ 1 \le j \le n\}$ is a basis of $U \otimes V$. Since $\mathbf{u} \otimes \mathbf{v} = \mathbf{u}_1 \otimes \mathbf{v}_1$ is an element of this basis, it is nonzero.

(b) Suppose that μ and λ are eigenvalues of S and T respectively. Then there exist nonzero vectors $\mathbf{u} \in U$ and $\mathbf{v} \in V$ such that $S(\mathbf{u}) = \mu\mathbf{u}$ and $T(\mathbf{v}) = \lambda\mathbf{v}$. By part (a), the vector $\mathbf{u} \otimes \mathbf{v}$ is nonzero. Furthermore, we have

$$(S \otimes T)(\mathbf{u} \otimes \mathbf{v}) = S(\mathbf{u}) \otimes T(\mathbf{v}) = (\mu\mathbf{u}) \otimes (\lambda\mathbf{v}) = \mu\lambda(\mathbf{u} \otimes \mathbf{v}).$$

Therefore $\mu\lambda$ is an eigenvalue of $S \otimes T$.

(c) Suppose S and T are diagonalizable. Then there exist bases $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of U and V, respectively, consisting of eigenvectors. Then $\{\mathbf{u}_i \otimes \mathbf{v}_j \ ; \ 1 \le i \le m,\ 1 \le j \le n\}$ is a basis of $U \otimes V$. By the proof of part (b), every element of this set is an eigenvector of $S \otimes T$. Thus $S \otimes T$ is diagonalizable.

12. **(5 pts)** Suppose that V is a finite-dimensional inner product space and that

$$T : V \to V$$

is a linear operator. Show that $\ker(T^*) = (\mathrm{Im}(T))^\perp$ and that $\mathrm{Im}(T^*) = (\ker(T))^\perp$.

Solution: For $\mathbf{v} \in V$, we have

$$\begin{aligned} \mathbf{v} \in \ker(T^*) &\iff T^*(\mathbf{v}) = 0 \iff \langle \mathbf{u}, T^*(\mathbf{v})\rangle = 0 \ \forall\ \mathbf{u} \in U \\ &\iff \langle T(\mathbf{u}), \mathbf{v}\rangle = 0 \ \forall\ \mathbf{u} \in U \iff \mathbf{v} \in (\mathrm{Im}\, T)^\perp. \end{aligned}$$

Thus $\ker(T^*) = (\mathrm{Im}\, T)^\perp$.

Now suppose $\mathbf{v} \in \mathrm{Im}(T^*)$. Then $\mathbf{v} = T^*(\mathbf{v}')$ for some $\mathbf{v}' \in V$. Thus, for all $\mathbf{u} \in \ker(T)$, we have

$$\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, T^*(\mathbf{v}')\rangle = \langle T(\mathbf{u}), \mathbf{v}'\rangle = \langle 0, \mathbf{v}'\rangle = 0.$$

Thus $\mathbf{v} \in (\ker(T))^\perp$. So $\mathrm{Im}(T^*) \subseteq (\ker(T))^\perp$. Now,

$$\begin{aligned} \dim \mathrm{Im}(T^*) &= \dim V - \dim \ker(T^*) \\ &= \dim V - \dim(\mathrm{Im}(T))^\perp \\ &= \dim \mathrm{Im}(T) \qquad (\text{since } (\mathrm{Im}(T))^\perp \oplus \mathrm{Im}(T) = V) \\ &= V - \dim \ker(T) \\ &= \dim(\ker(T))^\perp \qquad (\text{since } (\ker(T))^\perp \oplus \ker(T) = V). \end{aligned}$$

Therefore $\mathrm{Im}(T^*) = (\ker(T))^\perp$

Alternate solution for second equality: Replacing T by T^* in the first equality (and using that $(T^*)^* = T$) gives $\ker(T) = (\mathrm{Im}(T^*))^\perp$. Then, taking the orthogonal complement of both sides (and using the fact that $(W^\perp)^\perp = W$ for any subspace W of V), gives the second equality.

13. **(5 pts)** Consider the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$ given by

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = u_1 v_1 - \frac{1}{2} u_1 v_2 - \frac{1}{2} u_2 v_1 + u_2 v_2.$$

Let $T \in \operatorname{End}_{\mathbb{R}} \mathbb{R}^2$ be the linear operator whose matrix in the standard basis of $\mathbb{R}^2$ is

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

(a) Show that $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix} \right\}$ is an orthonormal basis of $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$.

(b) Considered as an operator on $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$, is T orthogonal?

(c) Considered as an operator on $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$, is T self-adjoint?

Solution:

(a) We check that

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = 1 - 0 - 0 + 0 = 1, \quad \left\langle \begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix} \right\rangle = \frac{1}{3}(1 - 1 - 1 + 4) = 1,$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix} \right\rangle = \frac{1}{\sqrt{3}} - \frac{1}{2} \frac{2}{\sqrt{3}} = 0.$$

Thus $\mathcal{B}$ is indeed an orthonormal basis.

(b) We now find the matrix of T in the orthonormal basis $\mathcal{B}$. We have

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \sqrt{3} \begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} = \sqrt{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}.$$

Since this matrix is not orthogonal, T is not orthogonal.

(c) Since the matrix $[T]_{\mathcal{B}}$ of T in the orthonormal basis $\mathcal{B}$ is symmetric, T is self-adjoint.