MidTerm - Solutions, MAT3141, Fall2010

1. The first solution:

From the definition of T we find that

$$T(v_1) = v_1 \text{ and } T(v_2) = -v_2.$$

Then

$$(2T + S)(v_1) = 2T(v_1) + S(v_1) = 2v_1 + (v_1 + v_2) = 3v_1 + v_2$$

and

$$(2T + S)(v_2) = 2T(v_2) + S(v_2) = 2(-v_2) + (v_1 - v_2) = v_1 - 3v_2.$$

We have

$$(2T + S)^{\circ 2}(v_1) = (2T + S)(3v_1 + v_2) = 3(3v_1 + v_2) + (v_1 - 3v_2) = 10v_1$$

and

$$(2T + S)^{\circ 2}(v_2) = (3v_1 + v_2) - 3(v_1 - 3v_2) = 10v_2$$

Therefore,

$$(2T + S)^{\circ 2}(v_1 + v_2) = 10v_1 + 10v_2.$$

The second solution:

The transformation matrices of T and S with respect to the basis $\{v_1, v_2\}$ are

$$M_T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad M_S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then

$$M_{2T+S} = 2M_T + M_S = \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$$

and

$$M_{(2T+S)^{\circ 2}} = (M_{2T+S})^2 = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$

Since the vector $v_1 + v_2$ has coordinates $(1, 1)$,

$$(1, 1) \cdot M_{(2T+S)^{\circ 2}} = (10, 10).$$

Hence, $(2T + S)^{\circ 2}(v_1 + v_2) = 10v_1 + 10v_2$.

2. (a)

We have for $p_1, p_2 \in V$

$$T(p_1 + p_2) = \int_0^1 (p_1 + p_2)'(x)x\ dx = \int_0^1 (p_1'(x) + p_2'(x))x\ dx =$$

$$\int_0^1 p_1'(x)x\ dx + \int_0^1 p_2'(x)x\ dx = T(p_1) + T(p_2)$$

and for $a \in \mathbb{R}$, $p \in V$

$$T(a \cdot p) = \int_0^1 (a \cdot p)'(x)x\ dx = a \cdot \int_0^1 p'(x)x\ dx = a \cdot T(p).$$

This shows that the map T is additive and homogeneous, hence, it is linear.

(b)

We have

$$T(1) = \int_0^1 (1)'x\ dx = \int_0^1 0 \cdot x\ dx = 0,$$

$$T(x) = \int_0^1 x' \cdot x\ dx = \int_0^1 x\ dx = \tfrac{1}{2}x^2|_0^1 = \tfrac{1}{2},$$

$$T(x^2) = \int_0^1 (x^2)' \cdot x\ dx = \int_0^1 2x^2\ dx = \tfrac{2}{3}x^3|_0^1 = \tfrac{2}{3},$$

$$T(x^3) = \int_0^1 (x^3)' \cdot x\ dx = \int_0^1 3x^3\ dx = \tfrac{3}{4}x^4|_0^1 = \tfrac{3}{4}.$$

Therefore, the transformation matrix is $(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4})^t$.

3. **(a)** Note that $\bar{1} = -\bar{1}$.

We have

$$x^5 + \bar{1} = x^2 \cdot (x^3 + \bar{1}) + (x^2 + \bar{1}),$$
$$x^3 + \bar{1} = x \cdot (x^2 + \bar{1}) + (x + \bar{1}),$$
$$x^2 + \bar{1} = (x + \bar{1})(x + \bar{1}).$$

Hence, $gcd(p_1(x), p_2(x)) = x + \bar{1}$.

(b)

We have

$$(x + \bar{1}) = p_1(x) + x \cdot (x^2 + \bar{1}) = p_1(x) + x \cdot (p_2(x) + x^2 p_1(x)) =$$

$$= (x^3 + \bar{1})p_1(x) + xp_2(x).$$

4. (a)

Note that all coefficients of U are divisible by $(x-1)$. Namely,

$$x^2-1=(x-1)(x+1),$$

$$x^2+x-2=(x-1)(x+2),$$

$$x^3-1=(x-1)(x^2+x+1).$$

Therefore,

$$U=(x-1)\cdot\begin{pmatrix} x+1 & 0 & 1 \\ x+2 & x^2+x+1 & 0 \end{pmatrix}=(x-1)\cdot U'.$$

Consider the matrix

$$\left(\begin{array}{ccc|cc} x+1 & 0 & 1 & 1 & 0 \\ x+2 & x^2+x+1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \end{array}\right)$$

We reduce its upper-left corner U' to the canonical form:
I. We perform $C_1-(x+1)C_3$:

$$\left(\begin{array}{ccc|cc} 0 & 0 & 1 & 1 & 0 \\ x+2 & x^2+x+1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ -(x+1) & 0 & 1 & & \end{array}\right)$$

II. We perform the Euclidean division

$$x^2+x+1=(x-1)\cdot(x+2)+3$$

and then $C_2-(x-1)C_1$:

$$\left(\begin{array}{ccc|cc} 0 & 0 & 1 & 1 & 0 \\ x+2 & 3 & 0 & 0 & 1 \\ \hline 1 & -(x-1) & 0 & & \\ 0 & 1 & 0 & & \\ -(x+1) & x^2-1 & 1 & & \end{array}\right)$$

III. We do $C_1 - \frac{x+2}{3} C_2$:

$$\left(\begin{array}{ccc|cc} 0 & 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ \hline \frac{x^2+x+1}{3} & -(x-1) & 0 & & \\ -\frac{x+2}{3} & 1 & 0 & & \\ -\frac{(x+1)(x^2+x+1)}{3} & x^2-1 & 1 & & \end{array}\right)$$

IV. Finally, we permute $C_1 \leftrightarrow C_3$:

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ \hline 0 & -(x-1) & \frac{x^2+x+1}{3} & & \\ 0 & 1 & -\frac{x+2}{3} & & \\ 1 & x^2-1 & -\frac{(x+1)(x^2+x+1)}{3} & & \end{array}\right)$$

Hence,

$$PUQ = P\Big((x-1)U'\Big)Q = (x-1) \cdot PU'Q = \begin{pmatrix} (x-1) & 0 & 0 \\ 0 & 3(x-1) & 0 \end{pmatrix}$$

The elementary divisors are $d_1 = (x-1)$, $d_2 = 3(x-1)$,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 1-x & \frac{x^2+x+1}{3} \\ 0 & 1 & -\frac{x+2}{3} \\ 1 & x^2-1 & -\frac{(x+1)(x^2+x+1)}{3} \end{pmatrix}$$