Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 3141 – Final Examination **Instructor: K. Zaynullin**

Last name: ______________________________

First name: ______________________________

Signature: ______________________________

Student number:

Read the following instructions:

- Not allowed: textbooks, course notes, electronic devices (calculators, telephones)
- Do not detach the pages of this examination.
- You may use the back of the pages as scrap paper for calculations, or to answer questions if you run out of space on the front side.

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1-5	6	7	8	9	10	11	Total
Mark								
Out of	10	5	7	13	7	5	8	55

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Questions 1-5 are short-answer questions: Provide only short answers.

1. Let V_1 and V_2 be vector spaces of dimensions n and m respectively. (2)

What is the dimension of $V_1^* \oplus (V_1 \otimes V_2)$?

`Answer:`

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2. Let A be a finite abelian group of order n considered as an $\mathbb{Z}$-module. (2)

How many torsion elements does A have ?

`Answer:`

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3. How many invertible elements does the ring $\mathbb{Z}/8\mathbb{Z}$ have ? (2)

`Answer:`

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4. Provide an example of a non-invertible 2×2 matrix M (2)

with coefficients in $\mathbb{Z}/6\mathbb{Z}$ such that $\det(M) \neq \bar{0}$.

`Answer:`

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5. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find the minimal polynomial of A. (2)

`Answer:`

Questions 6-11 are long-answer questions: You must give clear and complete solutions, with calculations, explanations and justifications. Make sure that your answer is clearly indicated.

6. Let A be an integral domain and let p and q be non-zero elements of A with $p \sim q$.

Show that p is irreducible if and only if q is irreducible. (5)

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7. Let V be a vector space over $\mathbb{Q}$ with basis $B = \{v_1, v_2, v_3\}$, and let $T \in End_{\mathbb{Q}}(V)$ be the linear operator satisfying

$$T(v_1) = v_2, \quad T(v_2) = v_3, \quad T(v_3) = v_1.$$

(a) Find the transformation matrix of T with respect to B. (1)

(b) Compute $T^{\circ 4}(10v_1 - 3v_2 + 7v_3)$. (2)

(c) It is known that the vector space V is a direct sum of two non-trivial T-invariant linear subspaces

$$V = V_1 \oplus V_2.$$

Find a basis (over $\mathbb{Q}$) for V_1 and V_2. (4)

(see the next page)

8. Let

$$A = \begin{pmatrix} 1-i & 1 & 0 \\ -1 & -1-i & 0 \\ 1 & 1 & -i \end{pmatrix}$$

be the matrix over $\mathbb{C}$.

(a) Find the characteristic polynomial of A. (1)

(b) Find the minimal polynomial of A. (2)

(c) Find the elementary divisors of A. (2)

(d) Find the rational canonical form of A. (4)

(e) Find the Jordan canonical form of A. (4)

(see the next page)

9. Let V denote the vector space of polynomials from $\mathbb{R}[x]$ of degree at most 2. Consider the map

$$b\colon V \times V \to \mathbb{R} \quad \text{given by} \quad (f,g) \mapsto \int_0^1 f(t)g(t)\,dt.$$

(a) Prove that b is a bilinear form. (3)

(b) Compute the matrix of b with respect to the basis $\{1, x, x^2\}$ of V. (4)

(see the next page)

10. Let $\{e_1, e_2\}$ denote the canonical basis of $\mathbb{R}^2$.

Prove that $e_1 \otimes e_2 - e_2 \otimes e_1$ can not be written in the form $u \otimes v$ with $u, v \in \mathbb{R}^2$. (5)

11. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map whose matrix with respect to the standard basis of $\mathbb{R}^3$ is

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Prove that T is an orthogonal linear operator. (2)

(b) Find an orthonormal basis B of $\mathbb{R}^3$ such that the transformation matrix of T with respect to B is block-diagonal with blocks of size 1 or 2. (3)

(c) Find the transformation matrix of T with respect to B. (3)

(this is the last page)