University of Ottawa

Department of Mathematics and Statistics

MAT 3120: Analysis III

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Midterm Test

October 19, 2010

1.

(a) [**2 pts**] State the definition of a *metric space*.

(b) [**3 pts**] Suppose d_1 and d_2 are two metrics on the same set X. Prove that the map $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = d_1(x,y) + d_2(x,y), \quad x, y \in X,$$

is also a metric on X.

2.

(a) [**1 pt**] State the definition of the *diameter* of a nonempty subset S of a metric space (X,d).

(b) [**1 pt**] State the definition of a *bounded* subset of a metric space (X,d).

(c) [**4 pts**] Suppose U and V are two bounded subsets of a metric space (X,d). Show that $U \cup V$ is bounded.

3. [**3 pts**] Let

$$\ell^\infty_{\mathbb{Q}} = \{(x_n)_{n=1}^\infty \in \ell^\infty \mid x_n \in \mathbb{Q} \ \forall\, n\}$$

be the metric subspace of ℓ^∞ consisting of sequences with rational terms (with the induced topology). Is $\ell^\infty_{\mathbb{Q}}$ a closed subspace of ℓ^∞? Remember to justify your answer.

4.

(a) [**1 pt**] State the definition of a *contraction mapping* on a metric space (X,d).

(b) [**2 pts**] Prove that a contraction mapping cannot have more than one fixed point.

5.

(a) [**1 pt**] State the definition of the *metric topology* on a metric space.

(b) [**1 pt**] State the definitions of a *cluster point* and a *closure point* in a metric space.

(c) [**2 pts**] Prove that every singleton in a metric space is closed in it.

6. [**5 pts**] Let X be an infinite set and let

$$\mathcal{T} = \{T \subseteq X \mid T = \varnothing \text{ or } T^{\complement} \text{ is finite}\}.$$

Show that $\mathcal{T}$ is a topology on X.