University of Ottawa

Department of Mathematics and Statistics

MAT 3120: Analysis III

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Midterm Test – Solutions

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1.

(a) **[2 pts]** State the definition of a *metric space*.

(b) **[3 pts]** Suppose d_1 and d_2 are two metrics on the same set X. Prove that the map $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) = d_1(x, y) + d_2(x, y), \quad x, y \in X,$$

is also a metric on X.

Solution:

(a) See the lecture notes.

(b) First, it is clear that $d : X \times X \to \mathbb{R}_+$ since d_1 and d_2 take nonnegative values (because they are metrics).

(M1) For all $x, y \in X$, we have

$$\begin{aligned} d(x, y) = 0 &\iff d_1(x, y) + d_2(x, y) = 0 \\ &\iff d_1(x, y) = 0 \text{ and } d_2(x, y) = 0 \iff x = y. \end{aligned}$$

The second implication follows from the fact that d_1 and d_2 take nonnegative values and the third follows from the fact that d_1 (or d_2) is a metric.

(M2) For all $x, y \in X$, we have

$$d(x, y) = d_1(x, y) + d_2(x, y) = d_1(y, x) + d_2(y, x) = d(y, x).$$

(M3) For all $x, y, z \in X$, we have

$$\begin{aligned} d(x, z) &= d_1(x, z) + d_2(x, z) \\ &\leq \big(d_1(x, y) + d_1(y, z)\big) + \big(d_2(x, y) + d_2(y, z)\big) \\ &= \big(d_1(x, y) + d_2(x, y)\big) + \big(d_2(x, y) + d_2(y, z)\big) \\ &= d(x, y) + d(y, z). \end{aligned}$$

2.

(a) [**1 pt**] State the definition of the *diameter* of a nonempty subset S of a metric space (X, d).
(b) [**1 pt**] State the definition of a *bounded* subset of a metric space (X, d).
(c) [**4 pts**] Suppose U and V are two bounded subsets of a metric space (X, d). Show that $U \cup V$ is bounded.

Solution:

(a) See the lecture notes.
(b) See the lecture notes.
(c) If U is empty, then $U \cup V = V$ is bounded. Similarly, if V is empty, then $U \cup V = U$ is bounded. Therefore, assume U and V are both nonempty. Since U and V are bounded, their diameters $\delta(U)$ and $\delta(V)$ are finite. Fix points $x_0 \in U$ and $y_0 \in V$. We claim the set

$$\{d(x, y) \mid x, y \in U \cup V\}$$

is bounded above by $\delta(U) + \delta(V) + d(x_0, y_0)$. Then it will follow that this set has a finite supremum and so $\delta(U \cup V)$ is finite, hence $U \cup V$ is bounded.

Consider arbitrary points $x, y \in U \cup V$. If $x, y \in U$, then

$$d(x, y) \leq \delta(U) \leq \delta(U) + \delta(V) + d(x_0, y_0),$$

and we're done. Similarly, if $x, y \in V$, then

$$d(x, y) \leq \delta(V) \leq \delta(U) + \delta(V) + d(x_0, y_0).$$

Now, if $x \in U$ and $y \in V$, then

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq \delta(U) + d(x_0, y_0) + \delta(V),$$

as desired. The case where $x \in V$ and $y \in U$ is analogous.

3. [**3 pts**] Let

$$\ell^\infty_{\mathbb{Q}} = \{(x_n)_{n=1}^\infty \in \ell^\infty \mid x_n \in \mathbb{Q} \ \forall\, n\}$$

be the metric subspace of ℓ^∞ consisting of sequences with rational terms (with the induced topology). Is $\ell^\infty_{\mathbb{Q}}$ a closed subspace of ℓ^∞? Remember to justify your answer.

Solution: No, this subspace is not closed. Consider the sequence $\{y_n\}$ whose nth term is

$$y_n = (p_n, 0, 0, \dots),$$

where p_n is the decimal expansion of π truncated after the nth decimal place. Then $\{y_n\}$ is clearly a sequence in ℓ^∞ since, for instance, the terms of each y_n are bounded by 4 (in absolute value). The sequence $\{y_n\}$ converges to $(\pi, 0, 0, \dots)$ in ℓ^∞, but this point is not in $\ell^\infty_{\mathbb{Q}}$.

4.

(a) **[1 pt]** State the definition of a *contraction mapping* on a metric space (X, d).
(b) **[2 pts]** Prove that a contraction mapping cannot have more than one fixed point.

Solution:

(a) See the lecture notes.
(b) Suppose $A : X \to X$ is a contraction mapping with contraction constant α (so $0 < \alpha < 1$) and x and y are fixed points of A. Then

$$0 \leq d(x, y) = d(Ax, Ay) \leq \alpha d(x, y) \implies d(x, y) = 0 \implies x = y.$$

5.

(a) **[1 pt]** State the definition of the *metric topology* on a metric space.
(b) **[1 pt]** State the definitions of a *cluster point* and a *closure point* in a metric space.
(c) **[2 pts]** Prove that every singleton in a metric space is closed in it.

Solution:

(a) See the lecture notes.
(b) See the lecture notes.
(c) Let (X, d) be a metric space and $x \in X$. We need to show that $\{x\}^{\complement}$ is open. Suppose $y \in \{x\}^{\complement}$. So $y \neq x$ and hence $r = d(x, y) > 0$. Then $B(y, r/2) \subseteq \{x\}^{\complement}$. Since our choice of y was arbitrary $\{x\}^{\complement}$ is open.

6. **[5 pts]** Let X be an infinite set and let

$$\mathcal{T} = \{T \subseteq X \mid T = \varnothing \text{ or } T^{\complement} \text{ is finite}\}.$$

Show that $\mathcal{T}$ is a topology on X.

Solution:

(T1) Since $X^{\complement} = \varnothing$ is finite, we have $\varnothing, X \in \mathcal{T}$.
(T2) Let $\mathcal{S} \subseteq \mathcal{T}$ be a collection of elements of $\mathcal{T}$. Then

$$\left(\bigcup_{T \in \mathcal{S}} T \right)^{\complement} = \bigcap_{T \in \mathcal{S}} T^{\complement},$$

which is finite since each T is (we can ignore any T that are empty). Hence $\bigcup_{T \in \mathcal{S}} T \in \mathcal{T}$.
(T3) Suppose $T_1, T_2 \in \mathcal{T}$. If either of T_1 or T_2 are empty, then $T_1 \cap T_2 = \varnothing \in \mathcal{T}$. If T_1 and T_2 are both nonempty, then

$$(T_1 \cap T_2)^{\complement} = T_1^{\complement} \cup T_2^{\complement},$$

which is finite. Hence $T_1 \cap T_2 \in \mathcal{T}$.