Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

MAT 3120: Analysis III

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Final Exam – December 21, 2010

Instructions:

(a) You have 3 hours to complete this exam.

(b) The number of points available for each question is indicated in square brackets.

(c) Unless otherwise indicated, you must justify your answers to receive full marks.

(d) All answers should be written in the examination booklets provided.

(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.

Good luck!

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1. (a) **[5 pts]** Define the following:
   (i) inner product space,
   (ii) normed space,
   (iii) metric space,
   (iv) toplogical space,
   (v) Banach space,
   (vi) Hilbert space.
   (b) **[1 pt]** If $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, define the associated norm (you do not need to prove it satisfies the axioms of a norm).
   (c) **[1 pt]** If $(X, \| \cdot \|)$ is a normed space, define the associated metric (you do not need to prove it satisfies the axioms of a metric).
   (d) **[1 pt]** Define the metric topology on a metric space (you do not need to prove it satisfies the axioms of a topology).

2. **[2 pts]** Let (X, d) be a metric space. Show that every convergent sequence in X is a Cauchy sequence.

3. **[2 pts]** Show that if A is a subset of a topological space B, then $\operatorname{bd}_B A = \operatorname{bd}_B A^{\complement}$.

4. **[4 pts]** Let $\{x_n\}, \{y_n\}$ be Cauchy sequences in an inner product space. Prove that $\{\langle x_n, y_n \rangle\}$ is a convergent sequence in $\mathbb{C}$.

5. **[2 pts]** Define

$$X = \{(x, \alpha x) \mid x \in \mathbb{R},\ \alpha \in \mathbb{Q}\}.$$

Show that X is a path-connected subset of $\mathbb{R}^2$.

6. (a) **[1 pt]** Define the ℓ^∞-norm on $\mathbb{C}^n$.
   (b) **[2 pts]** Verify that the ℓ^∞-norm does indeed satisfy the axioms of a norm.

7. **[3 pts]** Prove that a closed subset of a compact metric space is compact.

8. Suppose that $\| \cdot \|_1$ and $\| \cdot \|_2$ are two equivalent norms on a vector space X, and that Y is a subset of X.
   (a) **[2 pts]** Show that Y is closed with respect to $\| \cdot \|_1$ if and only if it is closed with respect to $\| \cdot \|_2$.
   (b) **[1 pt]** Show that Y is bounded with respect to $\| \cdot \|_1$ if and only if it is bounded with respect to $\| \cdot \|_2$.

9. **[2 pts]** Let (X, d) be a metric space and let $\mathcal{N}$ be the set of all Cauchy sequences in X (with or without limit). Define a relation $\sim$ on $\mathcal{N}$ by

$$\{x_n\} \sim \{y_n\} \iff d(x_n, y_n) \to 0, \quad \{x_n\}, \{y_n\} \in \mathcal{N}.$$

Prove that this defines an equivalence relation on $\mathcal{N}$.

10. Let $(X, \|\cdot\|)$ be a normed space.

(a) [**1 pt**] Show that $\|\cdot\|$ is a uniformly continuous map.

(b) [**2 pts**] Prove that if V is a finite-dimensional complex vector space, then the set

$$S = \{x \in V \mid \|x\|_\infty = 1\}$$

is compact. You may use the fact (proven in class) that

$$B = \{x \in V \mid \|x\|_\infty \leq 1\}$$

is compact.

11. [**3 pts**] Suppose X, Y and Z are normed vector spaces. If $A : Y \to Z$ and $B : X \to Y$ are bounded linear operators, show that $AB : X \to Z$ is also a bounded linear operator and that $\|AB\| \leq \|A\| \|B\|$. (You may assume that a composition of linear maps is linear.) Give an example that shows that this inequality can be strict (i.e. not equality).

12. Let n be a positive integer and suppose (X_i, d_i) is a metric space for each $i = 1, 2, \ldots, n$. Define $X = X_1 \times X_2 \times \cdots \times X_n$.

(a) [**2 pts**] Show that the map $d : X \times X \to \mathbb{R}_+$ defined by

$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i), \quad x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X,$$

is a metric on X.

(b) [**4 pts**] Show that if X_i is complete for each $i = 1, \ldots, n$, then X (with the metric defined above) is also complete.