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Mathématiques et de statistique
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MAT 3120: Analysis III

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Final Exam – Solutions – December 21, 2010

Instructions:

(a) You have 3 hours to complete this exam.

(b) The number of points available for each question is indicated in square brackets.

(c) Unless otherwise indicated, you must justify your answers to receive full marks.

(d) All answers should be written in the examination booklets provided.

(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.

Good luck!

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1. (a) **[5 pts]** Define the following:
   (i) inner product space,
   (ii) normed space,
   (iii) metric space,
   (iv) toplogical space,
   (v) Banach space,
   (vi) Hilbert space.

   (b) **[1 pt]** If $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, define the associated norm (you do not need to prove it satisfies the axioms of a norm).

   (c) **[1 pt]** If $(X, \|\cdot\|)$ is a normed space, define the associated metric (you do not need to prove it satisfies the axioms of a metric).

   (d) **[1 pt]** Define the metric topology on a metric space (you do not need to prove it satisfies the axioms of a topology).

Solution: See lecture notes.

2. **[2 pts]** Let (X, d) be a metric space. Show that every convergent sequence in X is a Cauchy sequence.

Solution: Suppose $\{x_n\}$ is a convergent sequence in X, with limit x. Choose $\epsilon > 0$. Then there exists $N > 0$ such that

$$n > N \implies \|x_n - x\| < \epsilon/2.$$

Then

$$m, n > N \implies \|x_n - x_m\| < \|x_n - x\| + \|x - x_m\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence.

3. **[2 pts]** Show that if A is a subset of a topological space B, then $\operatorname{bd}_B A = \operatorname{bd}_B A^{\complement}$.

Solution: Since $\left(A^{\complement}\right)^{\complement} = A$, we have

$$\begin{aligned}
\operatorname{bd}_B A &= \{x \in B \mid \forall\, \epsilon > 0,\ B(x,\epsilon) \cap A \neq \varnothing \text{ and } B(x,\epsilon) \cap A^{\complement} \neq \varnothing\} \\
&= \left\{x \in B \mid \forall\, \epsilon > 0,\ B(x,\epsilon) \cap \left(A^{\complement}\right)^{\complement} \neq \varnothing \text{ and } B(x,\epsilon) \cap A^{\complement} \neq \varnothing\right\} \\
&= \operatorname{bd}_B A^{\complement}.
\end{aligned}$$

4. **[4 pts]** Let $\{x_n\}, \{y_n\}$ be Cauchy sequences in an inner product space. Prove that $\{\langle x_n, y_n \rangle\}$ is a convergent sequence in $\mathbb{C}$.

Solution: Fix $\epsilon > 0$. Since Cauchy sequences are bounded, there exists an $M > 0$ such that $\|x_n\| < M$ and $\|y_n\| < M$ for all $n \in \mathbb{N}$. Choose $N_1 > 0$ such that

$$n, m > N_1 \implies \|x_n - x_m\| \le \epsilon/2M,$$

and $N_2 > 0$ such that

$$n, m > N_2 \implies \|y_n - y_m\| \le \epsilon/2M.$$

Set $N = \max\{N_1, N_2\}$. Then

$$\begin{aligned} n, m > N \implies |\langle x_n, y_n\rangle - \langle x_m, y_m\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y_m\rangle +_\langle x_n, y_m\rangle - \langle x_m, y_m\rangle| \\ &= |\langle x_n, y_n - y_m\rangle + \langle x_n - x_m, y_m\rangle| \\ &\le |\langle x_n, y_n - y_m\rangle| + |\langle x_n - x_m, y_m\rangle| \\ &\le \|x_n\|\,\|y_n - y_m\| + \|x_n - x_m\|\,\|y_m\| \\ &\le M\frac{\epsilon}{2M} + \frac{\epsilon}{2M}M \\ &= \epsilon. \end{aligned}$$

Thus $\{\langle x_n y_n\rangle\}$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, this sequence converges.

5. **[2 pts]** Define

$$X = \{(x, \alpha x) \mid x \in \mathbb{R},\ \alpha \in \mathbb{Q}\}.$$

Show that X is a path-connected subset of $\mathbb{R}^2$.

Solution: We have

$$X = \bigcup_{\alpha\in\mathbb{Q}} X_\alpha, \quad \text{where} \quad X_\alpha = \{(x, \alpha x) \mid x \in \mathbb{R}\} \text{ for } \alpha \in \mathbb{Q}.$$

Since each X_α is clearly path-connected (it is a straight line in $\mathbb{R}^2$), and $(0, 0) \in X_\alpha$ for all $\alpha \in \mathbb{Q}$, we have that X is connected by the theorem proven in class. (One could also explicitly describe a path between any two points by starting at the first point, moving to the origin in a straight line, then moving along a straight lie to the second point.)

6. (a) **[1 pt]** Define the ℓ^∞-norm on $\mathbb{C}^n$.

Solution: See lecture notes.

(b) **[2 pts]** Verify that the ℓ^∞-norm does indeed satisfy the axioms of a norm.

Solution: Since, for all $x = (x_1, \dots, x_n) \in \mathbb{C}^n$,

$$\|x\|_\infty = \max_{i=1}^{\infty} \|x_i\| = 0 \iff x_i = 0\ \forall\, i = 1, \dots, n \iff x = 0,$$

we see that (N1) is satisfied. Also, for $\alpha \in \mathbb{C}$,

$$\|\alpha x\|_\infty = \max_{i=1}^{\infty} \|\alpha x_i\| = |\alpha| \max_{i=1}^{n} \|x_i\| = |\alpha|\|x\|_\infty,$$

and so (N2) is satisfied. Finally, for $x, y \in \mathbb{C}^n$, we have

$$\|x + y\| = \max_{i=1}^{\infty} \|x_i + y_i\| \le \max_{i=1}^{\infty} (\|x_i\| + \|y_i\|) \le \max_{i=1}^{\infty} \|x_i\| + \max_{i=1}^{n} \|y_i\| = \|x\|_\infty + \|y\|_\infty.$$

7. **[3 pts]** Prove that a closed subset of a compact metric space is compact.

Solution: Let Y be a closed subset of a compact metric space (X, d). We will use the concept of sequential compactness (although one could also give a proof using the definition of compactness in terms of open covers). Suppose $\{x_n\}$ is a sequence in Y. Then $\{x_n\}$ is also a sequence in X and so there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to a point $x \in X$. But then $\{x_{n_k}\}_{k=1}^{\infty}$ is a sequence in Y converging to a point $x \in X$. Hence, since Y is closed, $x \in Y$. Then the subsequence converges in Y. It follows that Y is compact.

8. Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on a vector space X, and that Y is a subset of X.

(a) **[2 pts]** Show that Y is closed with respect to $\|\cdot\|_1$ if and only if it is closed with respect to $\|\cdot\|_2$.

(b) **[1 pt]** Show that Y is bounded with respect to $\|\cdot\|_1$ if and only if it is bounded with respect to $\|\cdot\|_2$.

Solution: Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, there exist positive numbers a and b such that

$$a\|x\|_1 \le \|x\|_2 \le b\|x\|_1 \quad \forall\, x \in X.$$

(a) Suppose Y is closed with respect to $\|\cdot\|_1$. Choose $x \in Y^{\complement}$. Then, since $Y^{\complement}$ is open with respect to $\|\cdot\|_1$, there exists an $\epsilon > 0$ such that

$$B_1 = \{y \in X \mid \|x - y\|_1 < \epsilon\} \subseteq Y^{\complement}.$$

In other words,

$$\|x - y\|_1 < \epsilon \implies y \in Y^{\complement}.$$

Then

$$\|x - y\|_2 < a\epsilon \implies \|x - y\|_1 \le \frac{1}{a}\|x - y\|_2 < \epsilon \implies y \in Y^{\complement},$$

and so

$$B_2 = \{y \in X \;\|\|x - y\|_2 < a\epsilon\} \subseteq Y^{\complement}.$$

Hence $Y^{\complement}$ is open with respect to $\|\cdot\|_2$ and so Y is closed with respect to $\|\cdot\|_2$. The converse is analogous (and follows from the fact that equivalence of norms is an equivalence relation).

(b) Suppose Y is bounded with respect to $\|\cdot\|_1$. Then there exists $M > 0$ such that

$$\|y\|_1 < M \quad \forall\, y \in Y.$$

Then

$$\|y\|_2 \le b\|y\|_1 \le bM \quad \forall\, y \in Y,$$

and so Y is bounded with respect to $\|\cdot\|_2$.

9. **[2 pts]** Let (X, d) be a metric space and let $\mathcal{N}$ be the set of all Cauchy sequences in X (with or without limit). Define a relation $\sim$ on $\mathcal{N}$ by

$$\{x_n\} \sim \{y_n\} \iff d(x_n, y_n) \to 0, \quad \{x_n\}, \{y_n\} \in \mathcal{N}.$$

Prove that this defines an equivalence relation on $\mathcal{N}$.

Solution: Since $d(x_n, x_n) = 0$ for all n, the relation is reflexive. Since $d(x_n, y_n) = d(y_n, x_n)$ for all n, the relation is symmetric. Now, suppose $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$ for Cauchy sequences $\{x_n\}, \{y_n\}, \{z_n\}$. Then

$$d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n) \to 0,$$

and so $\{x_n\} \sim \{z_n\}$. Hence the relation is transitive.

10. Let $(X, \|\cdot\|)$ be a normed space.

(a) **[1 pt]** Show that $\|\cdot\|$ is a uniformly continuous map.

Solution: For any $\epsilon > 0$, we trivially have

$$\|x - y\| < \epsilon \implies \|x - y\| < \epsilon \quad \forall\, x, y \in X.$$

Hence $\|\cdot\|$ is uniformly continuous.

(b) **[2 pts]** Prove that if V is a finite-dimensional complex vector space, then the set

$$S = \{x \in V \mid \|x\|_\infty = 1\}$$

is compact. You may use the fact (proven in class) that

$$B = \{x \in V \mid \|x\|_\infty \le 1\}$$

is compact.

Solution: Let $\{x_n\}$ be a sequence in S. Then, since $S \subseteq B$, $\{x_n\}$ is also a sequence in B. Since B is compact, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$, converging to some point $x \in B$. Now, since $\|\cdot\|_\infty$ is continuous, we have

$$\|x\|_\infty = \left\| \lim_{n\to\infty} \right\|_\infty = \lim_{n\to\infty} \|x_n\|_\infty = \lim_{n\to\infty} 1 = 1,$$

and so $x \in S$. Thus every sequence in S has a subsequence converging in S. So S is compact.

11. **[3 pts]** Suppose X, Y and Z are normed vector spaces. If $A : Y \to Z$ and $B : X \to Y$ are bounded linear operators, show that $AB : X \to Z$ is also a bounded linear operator and that $\|AB\| \le \|A\|\,\|B\|$. (You may assume that a composition of linear maps is linear.) Give an example that shows that this inequality can be strict (i.e. not equality).

Solution: For $x \in X$, we have

$$\|(AB)x\| = \|A(Bx)\| \le \|A\|\|Bx\| \le \|A\|\,\|B\|\,\|x\|.$$

This proves that AB is bounded and $\|AB\| \le \|A\|\,\|B\|$.

Define $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A(x,y) = (x, 2y), \quad B(x,y) = (2x, y), \quad (x,y) \in \mathbb{R}^2.$$

Then, for all $(x,y) \in \mathbb{R}^2$, we have

$$\|A(x,y)\| = \|(x,2y)\| = \sqrt{x^2+4y^2} \le \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2} = 2\|(x,y)\|.$$

Hence $\|A\| \le 2$. Since $\|A(0,1)\| = \|(0,2)\| = 2 = 2\|(0,1)\|$, we have $\|A\| \ge 2$, and so $\|A\| = 2$. The same argument, with x and y reversed shows that $\|B\| = 2$. Now,

$$(AB)(x,y) = A(2x,y) = (2x,2y),$$

and it is easy to check (by the same method as above) that $\|AB\| = 2$. Thus

$$\|AB\| = 2 \ne 4 = \|A\|\,\|B\|.$$

12. Let n be a positive integer and suppose (X_i, d_i) is a metric space for each $i = 1, 2, \dots, n$. Define $X = X_1 \times X_2 \times \cdots \times X_n$.

(a) **[2 pts]** Show that the map $d : X \times X \to \mathbb{R}_+$ defined by

$$d(x,y) = \sum_{i=1}^n d_i(x_i, y_i), \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in X,$$

is a metric on X.

Solution: For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in X$, we have

$$d(x,y) = 0 \iff \sum_{i=1}^n d_i(x_i,y_i) = 0 \iff d_i(x_i,y_i) = 0\ \forall\ i \iff x_i = x_i\ \forall\ i \iff x = y.$$

Thus (M1) is satisfied. Since, for all $x, y \in X$,

$$d(x,y) = \sum_{i=1}^n d_i(x_i,y_i) = \sum_{i=1}^n d_i(y_i,x_i) = d(y,x),$$

we see that (M2) is also satisfied. Finally, for $x, y, z \in X$, we have

$$d(x,z) = \sum_{i=1}^n d_i(x_i,z_i) \le \sum_{i=1}^n \left(d_i(x_i,y_i) + d_i(y_i,z_i)\right)$$

$$= \sum_{i=1}^n d_i(x_i,y_i) + \sum_{i=1}^n d_i(y_i,z_i) = d(x,y) + d(y,z).$$

Therefore (M3) is satisfied.

(b) **[4 pts]** Show that if X_i is complete for each $i = 1, \dots, n$, then X (with the metric defined above) is also complete.

Solution: Suppose $\{x_m\}$ is a Cauchy sequence in X with $x_n = (x_{m1}, x_{m2}, \dots, x_{mn})$ for all $m \in \mathbb{N}$. Choose $\epsilon > 0$. Then there exists $N > 0$ such that

$$m, k > N \implies \|x_m - x_k\| < \epsilon.$$

Thus for each $1 \le i \le n$, we have

$$m, k > N \implies d_i(x_{mi}, x_{ki}) \le \sum_{j=1}^{n} d_j(x_{mj}, x_{kj}) = \|x_m - x_k\| < \epsilon.$$

Therefore, each $\{x_{mi}\}_{m=1}^{\infty}$ is a Cauchy sequence in X_i. Since each X_i is complete, this sequence converges to some y_i. Let $y = (y_1, \dots, y_n) \in X$. Since $x_{mi} \to y_i$ as $m \to \infty$, for each $1 \le i \le n$ we can choose an $N_i > 0$ such that

$$m > N_i \implies d_i(x_{mi}, y_i) < \epsilon/n.$$

Set $N = \max\{N_1, \dots, N_n\}$. Then

$$m > N \implies d(x_m, y) = \sum_{j=1}^{n} d_j(x_{mj}, y_j) \le \sum_{j=1}^{n} \frac{\epsilon}{n} = \epsilon.$$

Thus $x_m \to y$. Thus we have shown that every Cauchy sequence in X converges in X. So X is complete.