uOttawa
Department of Mathematics and Statistics
Département de mathématiques et statistique

Analysis III – Mat 3120
Mid-Term Exam — June 4, 2008

Time: 1 hour 50 minutes.
Attempt all questions in (1)–(3).
Only attempt problem (4⋆) after you have done everything else.
Total number of marks: 27 (+ 3 bonus marks).

(1) (a) [**1 point**] What does it mean that a pair (X, d) is a metric space? Give a definition.

(b) Give the definitions of

(i) [**1 point**] an open subset of a metric space,

(ii) [**1 point**] a closed subset of a metric space,

(iii) [**1 point**] an interior point and the interior of a set,

(iv) [**1 point**] a closure point and the closure of a set.

(c) [**2 points**] Some topologists call a closed subset F of a metric space X a *"canonical closed subset"* if

$$F = \mathrm{cl}\,(\mathrm{Int}\, F)\,.$$

Give an example of a closed subset in a metric space which is not a "canonical closed subset," with necessary explanations.

(d) [**2 points**] Let X be a metric space whose metric takes its values in $\{0, 1\}$. Prove that every subset of X is a "canonical closed subset."

[Continued on the next page....]

(2) (a) [**3 points**] Recall that a subset A of a metric space X is (everywhere) dense in X if every point of X is a closure point for A.
Let X be a metric space, and let $Y, Z \subseteq X$ be subsets of X, such that $Z \subseteq Y$. Assume that Z is dense in Y, and Y is dense in X. Prove that Z is dense in X.

(b) [**1 point**] State a definition of a continuous mapping between two metric spaces.

(c) [**3 points**] Prove that a continuous image of an everywhere dense subset under a surjective map is everywhere dense in the image. In other words, let $f: X \to Y$ be a continuous map onto, and let $A \subseteq X$ be an everywhere dense subset of X. Prove that $f(A)$ is everywhere dense in Y.

(d) [**2 points**] Let $f: X \to Y$ be a continuous mapping between two metric spaces. Assume a subset $B \subseteq Y$ is everywhere dense in Y. Is it true that the inverse image $f^{-1}(B)$ is everywhere dense in X? If yes, give a proof. If no, construct a counter-example.

(3) (a) [**2 points**] State the definition of a disconnected metric space; of a connected metric space.

(b) [**1 point**] State the definition of a path-connected metric space.

(c) [**1 point**] Describe, without giving any proofs, what is the relationship between the classes of connected and of path-connected metric spaces.

(d) Let X be a metric space, and let A be an everywhere dense subset of X.

(i) [**3 points**] Suppose A is connected. Prove that X is connected as well.

(ii) [**1 point**] Now assume that A is disconnected. Is X necessarily disconnected as well? If yes, give a proof; if no, point at a counter-example.

(iii) [**1 point**] Finally, assume that A is path-connected. Does X have to be path-connected? Explain by referring to known results from lectures, without giving proofs.

(4) [**★ bonus question — 3 points**] We have seen in a lecture that every sequence (x_n) of elements of a metric space X satisfying the condition $\sum_{i=1}^{\infty} d(x_n, x_{n+1}) < \infty$ is a Cauchy sequence. Show that the converse statement fails for every metric space X admiting a non-trivial Cauchy sequence (that is, one which is not eventually constant).

(*Hint:* it makes sense to begin by clearly stating what the converse statement is!)

[End of the exam questions]