Analysis III – Mat 3120

June 11, 2008

Solutions to Mid-Term Exam

(1) (a) [**1 point**] What does it mean that a pair (X, d) is a metric space? Give a definition.

$\triangleleft$ This statement means that X is a non-empty set equipped with a metric d, that is, a mapping $d\colon X \times X \to \mathbb{R}$ satisfying three certain axioms (that need to be stated). $\triangleright$

(b) Give the definitions of

(i) [**1 point**] an open subset of a metric space,

$\triangleleft$ A subset V of a metric space X is open in X if for every $x \in V$ there is an $\epsilon > 0$ such that the open ball $B_\epsilon(x) \subseteq V$, where $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$. $\triangleright$

(ii) [**1 point**] a closed subset of a metric space,

$\triangleleft$ A subset F of a metric space X is closed in X if the complement $F^c = X \setminus F$ is open in X. $\triangleright$

(iii) [**1 point**] an interior point and the interior of a set,

$\triangleleft$ A point x of a subset A of a metric space X is said to be an interior point of A in X if for some $\epsilon > 0$ the open ball $B_\epsilon(x)$ is contained in A. The interior of a set A as above, denoted $\operatorname{Int}_X A$ or simply $\operatorname{Int} A$ when there is no confusion, is the set of all interior points of A. $\triangleright$

(iv) [**1 point**] a closure point and the closure of a set.

$\triangleleft$ A point x of a metric space X is a closure point for a subset $A \subseteq X$ if every neighbourhood of x meets A, that is, for all $\epsilon > 0$ the set $B_\epsilon(x) \cap A$ is non-empty. The closure of A in X, denoted $\operatorname{cl}_X A$ or simply $\operatorname{cl} A$ when no confusion can arise, is the set of all closure points of A in X. $\triangleright$

(c) [**2 points**] Some topologists call a closed subset F of a metric space X a *"canonical closed subset"* if

$$F = \operatorname{cl}\,(\operatorname{Int} F)\,.$$

Give an example of a closed subset in a metric space which is not a "canonical closed subset," with necessary explanations.

◁ The simplest example possible would be any singleton $\{x\}$ where the point x is not isolated, that is, the set $F = \{x\}$ is not open. This is the case, for instance, if $x \in \mathbb{R}$. In such a case, F is closed, but is not a "canonical closed subset:" indeed, since every singleton in every metric space is closed, we have

$$\operatorname{Int}\operatorname{cl}\{x\} = \operatorname{Int}\{x\} = \emptyset \neq \{x\}.$$

Of course numerous other examples of this kind are also possible. ▷

(d) [**2 points**] Let X be a metric space whose metric takes its values in $\{0, 1\}$. Prove that every subset of X is a "canonical closed subset."

◁ Let $A \subseteq X$ be arbitrary. Since every subset A of a metric space with a 0-1 metric is at the same time open and closed, we have

$$\operatorname{Int}\operatorname{cl} A = \operatorname{Int} A = A,$$

and so A is a "canonical closed subset" of X. ▷

(2) (a) [**3 points**] Recall that a subset A of a metric space X is (everywhere) dense in X if every point of X is a closure point for A.
Let X be a metric space, and let $Y, Z \subseteq X$ be subsets of X, such that $Z \subseteq Y$. Assume that Z is dense in Y, and Y is dense in X. Prove that Z is dense in X.

◁ Let $x \in X$ and $\epsilon > 0$ be arbitrary. Since Y is dense in X by hypothesis, the set $B^X_{\epsilon/2}(x) \cap Y$ is non-empty, where the open ball is formed in X. Choose a point $y \in B_{\epsilon/2}(x) \cap Y$. Since Z is dense in Y, the set $B^Y_{\epsilon/2}(y) \cap Z$ is non-empty, where the ball is formed in Y. Choose a point $z \in B^Y_{\epsilon/2}(y) \cap Z$. Now one has, using the triangle inequality,

$$d(x, z) \leq d(x, y) + d(y, z) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We conclude:

$$z \in B^X_\epsilon(x),$$

and since x and ϵ were arbitrary, this means that Z is everywhere dense in X, as required. ▷

(b) [**1 point**] State a definition of a continuous mapping between two metric spaces.

◁ A mapping f from a metric space $X = (X, d_X)$ to a metric space $Y = (Y, d_Y)$ is continuous if it is continuous at every point of X, that is, if for each $x \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ with the property that whenever $y \in X$ and $d_X(x, y) < \delta$, one has $d_Y(f(x), f(y)) < \epsilon$. ▷

(c) **[3 points]** Prove that a continuous image of an everywhere dense subset under a surjective map is everywhere dense in the image. In other words, let $f\colon X \to Y$ be a continuous map onto, and let $A \subseteq X$ be an everywhere dense subset of X. Prove that $f(A)$ is everywhere dense in Y.

$\triangleleft$ We will verify the definition of an everywhere dense subset. Let $y \in Y$ and $\epsilon > 0$ be arbitrary. Since f is surjective, there is a point $x \in X$ with $f(x) = y$. Since f is continuous at x, there exists $\delta > 0$ with the property that whenever $y \in X$ and $d_X(x, y) < \delta$, one has $d_Y(f(x), f(y)) < \epsilon$. Since by assumption A is everywhere dense in X, one can find a point $a \in A$ with the property $a \in B_\delta(x)$. Now one has $d_Y(f(a), y) = d_Y(f(a), f(x)) < \epsilon$, and since $f(a) \in f(A)$, one concludes: the set $f(A)$ is everywhere dense in Y, as required. $\triangleright$

(d) **[2 points]** Let $f\colon X \to Y$ be a continuous mapping between two metric spaces. Assume a subset $B \subseteq Y$ is everywhere dense in Y. Is it true that the inverse image $f^{-1}(B)$ is everywhere dense in X? If yes, give a proof. If no, construct a counter-example.

$\triangleleft$ The answer is negative: the statement is in general false. Here is one possible counter-example, about the simplest possible. Let $X = \{*\}$ be a singleton metric space, with a trivial distance, and let $Y = \mathbb{R}$ be the real line, with the usual metric. Define $f\colon X \to Y$ by the condition $f(*) = 0$, and set $B = \mathbb{R} \setminus \mathbb{Q}$ to be equal to the subset of irrational numbers. Then B is everywhere dense in $\mathbb{R}$, yet the inverse image of B under f is empty, and thus not everywhere dense in X:

$$f^{-1}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset.$$

$\triangleright$

(3) (a) **[2 points]** State the definition of a disconnected metric space; of a connected metric space.

$\triangleleft$ A metric space X is called disconnected if there exist two disjoint non-empty open subsets V and U of X whose union is X:

$$U \cap V = X.$$

A metric space X is called connected if it is not disconnected. In other words, X is connected if, whenever $X = U \cap V$, where U, V are disjoint open subsets of X and $U \neq \emptyset$, one necessarily has $V = \emptyset$. $\triangleright$

(b) **[1 point]** State the definition of a path-connected metric space.

$\triangleleft$ A metric space X is path-connected if for every two points $x, y \in X$

there exists a continuous mapping $p\colon [0,1] \to X$ (called a path between x and y) with the properties $p(0) = x$ and $p(1) = y$. ▷

(c) **[1 point]** Describe, without giving any proofs, what is the relationship between the classes of connected and of path-connected metric spaces.

◁ Every path-connected metric space is connected, but not vice versa: there exist examples of connected metric spaces which are not path-connected. ▷

(d) Let X be a metric space, and let A be an everywhere dense subset of X.

(i) **[3 points]** Suppose A is connected. Prove that X is connected as well.

◁ Let X be represented as a union of two disjoint open subsets U and V. Suppose U is non-empty. Since A is everywhere dense in X, the intersection $U' = U \cap A$ is non-empty as well. The sets U' and $V' = V \cap A$ are open in A, disjoint between themselves, and clearly

$$U' \cup V' = A.$$

Since A is assumed to be connected, we conclude: $V' = \emptyset$. Because of everywhere density of A in X, this necessarily means $V = \emptyset$, and we are done. ▷

(ii) **[1 point]** Now assume that A is disconnected. Is X necessarily disconnected as well? If yes, give a proof; if no, point at a counter-example.

◁ Under this assumption, the metric space X need not be disconnected. Among the best known examples is the real line $X = \mathbb{R}$, which is connected, yet contains as an every dense subset the metric space $\mathbb{Q}$ of rational numbers, which is disconnected (and even totally disconnected). ▷

(iii) **[1 point]** Finally, assume that A is path-connected. Does X have to be path-connected? Explain by referring to known results from lectures, without giving proofs.

◁ The answer is No, again. We have seen a counter-example in the lectures: our example X of a connected non path-connected metric subspace of the plane contained an everywhere dense path-connected part X_1 (the graph of $\cos \frac{1}{x}$). ▷

(4) [★ **bonus question — 3 points**] We have seen in a lecture that every sequence (x_n) of elements of a metric space X satisfying the condition $\sum_{i=1}^{\infty} d(x_n, x_{n+1}) < \infty$ is a Cauchy sequence. Show that the converse statement fails for every metric space X admiting a non-trivial Cauchy sequence (that is, one which is not eventually constant).

◁ Here is the converse statement:

Let X be a metric space. Then every Cauchy sequence $(x_n)_{n=1}^{\infty}$ of elements of X satisfies the condition $\sum_{i=1}^{\infty} d(x_n, x_{n+1}) < \infty$.

Of course this property is satisfied in a metric space whose only Cauchy sequences are the virtually constant ones (e.g. in a metric space equipped with a 0-1 valued metric).

Let us prove that this is the only case where the above conclusion holds. Let X be a metric space admitting a Cauchy sequence (x_n) that is not virtually constant. This means: for each N, there is $n \geq N$ such that $x_N \neq x_n$.

Since (x_n) is also Cauchy, one can find moreover an $n \geq N$ so that x_n is distinct from all the elements $x_1, x_2, \ldots, x_N$. Indeed, it is enough to take as $\epsilon > 0$ the smallest distance from x_N to any element in $\{x_1, x_2, \ldots, x_N\} \setminus \{x_N\}$, and notice that if n is large enough, then x_n is at a distance $< \epsilon$ from x_N and yet different from x_N.

Now, by proceeding to a recursively chosen subsequence, one can assume without loss in generality that (x_n) has the property that all the elements x_n, $n = 1, 2, 3, \ldots$, are pairwise distinct.

In order to transform (x_n) into a Cauchy sequence which fails the summability condition

$$\sum_{i=1}^{\infty} d(x_n, x_{n+1}) < \infty, \tag{0.1}$$

we will do the following. For each n, denote

$$k_n = \left\lfloor \frac{1}{d(x_n, x_{n+1})} \right\rfloor + 1.$$

Now define a new sequence, as follows: let it begin with

$$a_1, a_2, a_1, a_2, \ldots \ (k_1 \text{ repetitions of the pair}),$$

followed by

$$a_2, a_3, a_2, a_3, \ldots \ (k_2 \text{ repetitions}),$$

and so forth: every pair a_n, a_{n+1} will be repeated k_n times.

Notice that, by the choice of k_n, one has for every n

$$\underbrace{d(a_n, a_{n+1}) + d(a_n, a_{n+1}) + \ldots + d(a_n, a_{n+1})}_{k_n \text{ times}} > 1.$$

The new sequence — denote it (y_n) — is easily seen to be Cauchy, by an application of the triangle inequality.

Yet the sum of all distances $d(y_n, y_{n+1})$ will be at least as large as the sum of the series

$$1 + 1 + 1 + 1 + \ldots = \infty.$$

$\triangleright$