uOttawa

Department of Mathematics
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Département de mathématiques
et statistique

Analysis III – Mat 3120

Final Exam — June 21, 2008

Time: 3 hours.
Attempt ALL three questions.
Each question is worth 11 marks.
Total number of marks: 33
This is a closed book exam.
No electronic devices are allowed.

(1) (a) Let X be a metric space. State the definitions of an open subset of X and of a closed subset of X. **[1 mark]**

(b) State what is a closure point of a set in a metric space, and formulate (without a proof) a criterion for a set to be closed in a metric space in the language of closure points. **[1 mark]**

(c) Let X be a metric space, and let Y be a metric subspace of X. Further assume that a subset $F \subseteq X$ is closed in X. Prove that the intersection $F \cap Y$ is closed in the metric space Y. **[2 marks]**

(d) Let Z be a subset of a metric space X. Prove that Z is closed in X if and only if every point $x \in X$ has a neighbourhood, V, with the property that $Z \cap V$ is closed in V. **[2 marks]**

(e) A subset Z of a metric space X is called *locally closed* if every point $x \in Z$ has a neighbourhood, V, in X with the property that $Z \cap V$ is closed in V. Give an example of a locally closed subset of $\mathbb{R}$ which is not closed, with explanations. **[2 marks]**

(f) Prove that a subset Z of a metric space X is locally closed in X if and only if there exist an open subset $U \subseteq X$ and a closed subset $F \subseteq X$ such that $Z = U \cap F$. **[3 marks]**

[Continued overleaf....]

(2) (a) State the definition of a normed space, and of a metric associated to a norm. **[1 mark]**

(b) Let E be a normed space, and let F be a vector subspace of E. Assume that the interior of F in E is non-empty. Prove that $E = F$. **[2 marks]**

(c) For every natural n, let $\mathscr{P}_n[0,1]$ denote the vector space of all polynomial functions of degree $\leq n$ on the closed unit interval. What is the vector space dimension of $\mathscr{P}_n[0,1]$? Explain. **[1 mark]**

(d) Deduce from the observation made in problem (2c) that the vector space $\mathscr{P}_n[0,1]$ is closed in the normed space $C[0,1]$, and state all the results from the course that you are using. **[2 marks]**

(e) Deduce from the above results that the space $\mathscr{P}[0,1]$ is meager in $C[0,1]$, with all the necessary explanations and references to results used. **[2 marks]**

(f) Deduce from the Baire Category Theorem that not every continuous function on $[0,1]$ is polynomial, and state all the results you are using. **[1 mark]**

(g) Modify the above argument to establish the following: a Banach space E never has countably infinite dimension (as a vector space). In other words, a vector basis for E is either finite or uncountable. **[2 marks]**

[Continued on the opposite page....]

(3) (a) Give a complete definition of what it means that a metric space X is totally bounded, including all the necessary auxiliary concepts. **[1 mark]**

(b) Define what it means that a metric space X is separable. **[1 mark]**

(c) Prove that every totally bounded metric space is separable. **[2 marks]**

(d) Let $\mathscr{F}$ be a family of mappings from a metric space X to a metric space Y. Define what it means that the family $\mathscr{F}$ is uniformly equicontinuous. **[1 mark]**

(e) Let X be a metric space. For every $x \in X$ denote d_x a function from X to $\mathbb{R}$ defined by letting $d_x(y) = d(x, y)$. Denote by $\mathscr{D}_X$ the family of all distance functions on X:

$$\mathscr{D}_X = \{d_x \colon x \in X\}.$$

(i) Show that the family $\mathscr{D}_X$ is uniformly equicontinuous. **[1 mark]**

(ii) Let now K be a compact metric space. Denote by $C(K)$ the vector space of all continuous real-valued functions on K equipped with the uniform norm $\| f \|_\infty = \max_{x\in K} |f(x)|$ (and the corresponding metric). Prove that the mapping

$$K \ni x \mapsto d_x \in C(K),$$

associating to every point $x \in K$ the corresponding distance function, is an isometric embedding, that is, for every $x, y \in X$ one has $d(x, y) = \| d_x - d_y \|_\infty$. **[3 marks]**

(iii) Conclude (with explanations) that, under the above assumptions, the family $\mathscr{D}_K$ of all distance functions on K is closed in $C(K)$. **[1 mark]**

(iv) Let again K be compact. Prove that the family $\mathscr{D}_K$ of distance functions on K is uniformly bounded in $C(K)$. **[1 mark]**

[End of the exam questions]