uOttawa

Department of Mathematics
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Département de mathématiques
et statistique

Analysis III – Mat 3120

Mid-Term Exam — October 24, 2006

Time: 1 hour 20 minutes.
Attempt ALL questions.
Total number of marks: 25

(1) (a) [**1 point**] What does it mean that a pair (X, d) is a metric space? Give a definition.

(b) Let X be a set, and $f : X \to \mathbb{R}$ a function. Define a function of two arguments by

$$d_f(x, y) = |f(x) - f(y)|.$$

(i) [**2 points**] Is d_f necessarily a metric on X? Which conditions hold and which do not? Explain.

(ii) [**1 point**] What is a restriction on f necessary and sufficient for d_f to be a metric? Give a proof.

(c) What does it mean that a sequence (x_n) of points in a metric space is

(i) [**1 point**] convergent?

(ii) [**1 point**] Cauchy?

(d) [**1 points**] State the definition of a compact metric space.

(e) [**3 points**] We know that every metric space is a subspace of a complete metric space. Is it true that every metric space is a metric subspace of a *compact* metric space? Explain.

[Continued on the next page....]

(2) (a) **[1 point]** Give a definition of a continuous mapping between two metric spaces.

(b) **[3 points]** Show that in an arbitrary metric space continuous real-valued functions *separate points*, that is, for every $x, y \in X$, if $x \neq y$, then there exists a continuous real-valued function $f\colon X \to \mathbb{R}$ such that $f(x) \neq f(y)$.

(c) **[1 point]** State what it means that a subset A of a metric space X is everywhere dense in X.

(d) **[2 points]** Let $f\colon X \to Y$ be a continuous mapping between two metric spaces. Assume a subset $B \subseteq Y$ is everywhere dense in Y. Is it true that the inverse image $f^{-1}(B)$ is everywhere dense in X? If yes, give a proof. If no, construct a counter-example.

(3) (a) **[2 points]** State the definition of a disconnected metric space; of a connected metric space.

(b) **[2 points]** Prove that a continuous image of a connected metric space is connected.

(c) **[2 points]** Is it true or false that a continuous image of a disconnected space is disconnected? If yes, give a proof. If no, construct a counter-example.

(d) **[2 points]** A metric space X is called *totally disconnected* if every metric subspace Y of X, containing more than one point, is disconnected. Prove that the space $\mathbb{Q}$ of rational numbers with the usual distance is totally disconnected.

[End of the exam questions]