Analysis III – Mat 3120

♯ 19 – October 24, 2006

Solutions to the Mid-Term Exam

(1) (a) **[1 point]** *What does it mean that a pair (X, d) is a metric space? Give a definition.*

Cf. Definitions 3.1 and 3.5.

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(b) *Let X be a set, and $f\colon X \to \mathbb{R}$ a function. Define a function of two arguments by*

$$d_f(x, y) = |f(x) - f(y)|.$$

(i) **[2 points]** *Is d_f necessarily a metric on X? Which conditions hold and which do not? Explain.*

(ii) **[1 point]** *What is a restriction on f necessary and sufficient for d_f to be a metric? Give a proof.*

Cf. solution to problem 1 of Assignment 2.

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(c) *What does it mean that a sequence (x_n) of points in a metric space is*

(i) **[1 point]** *convergent?*

See Definition 4.16.

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(ii) **[1 point]** *Cauchy?*

Cf. Definition 8.1.

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(d) **[1 points]** *State the definition of a compact metric space.*

Definition 11.2.

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(e) **[3 points]** *We know that every metric space is a subspace of a complete metric space. Is it true that every metric space is a metric subspace of a* compact *metric space? Explain.*

No. In fact, it is not difficult to prove that every metric subspace Y of a totally bounded metric space X is again totally bounded. Indeed, let

$\epsilon > 0$ be arbitrary. Choose a finite $\epsilon/2$-net, Z, in X. For every $z \in Z$ with the property $B_{\epsilon/2}(z) \cap Y \neq \emptyset$ choose a point $y_z \in B_{\epsilon/2}(z) \cap Y$. In this way, we obtain a finite set Z' of points of the form y_z. Clearly, $Z' \subseteq Y$. We claim Z' is an ϵ-net for Y. Indeed, let $y \in Y$ be arbitrary. Since Z is an $\epsilon/2$-net for X, there is a $z \in Z$ with the property $d(z, y) < \epsilon/2$. In particular, the ball $B_{\epsilon/2}(z)$ meets Y, and so there is a point $y_z \in Z'$ at a distance $< \epsilon/2$ from z. By the triangle inequality,

$$d(y, y_z) \leq d(y, z) + d(z, y_z) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We have shown that for every ϵ, the subspace Y admits a finite ϵ-net, and so Y is totally bounded.

As a consequence, if Y is a non-totally bounded metric space, it is not a metric subspace of a compact space. (Moreover, one can prove that total boundedness of a metric space Y is a necessary and sufficient condition for Y to be a metric subspace of a compact space.)

One can give an alternative solution. Let Y be a metric space containing a sequence of elements (x_i) with the property that for some $\epsilon > 0$, one has $d(x_i, x_j) \geq \epsilon$ for all $i \neq j$. (These are precisely non-totally bounded spaces, as we have seen in the lectures.) The sequence (x_i) has the same property in X, and so contains no Cauchy subsequence, and no convergent subsequence either. For instance, $\mathbb{R}$ and $\mathbb{Z}$ with their usual distances, as well as an infinite discrete metric space, are not metric subspaces of compact metric spaces.

And here is the simplest argument which is obtained by simply putting together known results from our course. (Which is of course OK because this problem never appeared in the course before.) If a complete metric space X is a subspace of a compact metric space Y, then X is closed in Y, hence compact. Therefore, any complete non-compact metric space, such as $\mathbb{R}$, gives an example of a metric space that does not sit inside a compact metric space as a metric subspace.

Requirements, as usual: everything should be carefully explained starting from scratch.

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(2) (a) [**1 point**] *Give a definition of a continuous mapping between two metric spaces.*

Definitions 7.2 and 7.4, although any of the equivalent forms of the definition, as established in Lecture 7, would also do.

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(b) [**3 points**] *Show that in an arbitrary metric space continuous real-valued functions* separate points, *that is, for every $x, y \in X$, if $x \neq y$, then there* *exists a continuous real-valued function* $f\colon X \to \mathbb{R}$ *such that* $f(x) \neq f(y)$.

Let $a, b \in X$ be such that $a \neq b$. The function $f\colon X \to \mathbb{R}$, defined by

$$f(x) = d(a, x),$$

is continuous. Indeed, it is 1-Lipschitz:

$$f(x) - f(y) = d(a, x) - d(y, a) \leq d(x, y)$$

(a different way to write the triangle inequality), and consequently

$$|f(x) - f(y)| \leq d(x, y),$$

therefore for a given $\epsilon > 0$, by setting $\delta = \epsilon$, we have:

$$\forall x, y \in X,\ d(x, y) < \delta \Rightarrow |f(x) - f(y)| \leq d(x, y) < \delta = \epsilon.$$

At the same time, $f(a) = d(a, a) = 0 < d(a, b) = f(b)$, and so we conclude: continuous real-valued functions separate points of X.

Here again I expect full statements and proofs.

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(c) [**1 point**] *State what it means that a subset* A *of a metric space* X *is everywhere dense in* X.

Definition 5.7.

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(d) [**2 points**] *Let* $f\colon X \to Y$ *be a continuous mapping between two metric spaces. Assume a subset* $B \subseteq Y$ *is everywhere dense in* Y. *Is it true that the inverse image* $f^{-1}(B)$ *is everywhere dense in* X*? If yes, give a proof. If no, construct a counter-example.*

Here I meant to require that f be a surjection, which makes the problem slightly more challenging. But even then, the answer is, No, this is not true in general. For instance, denote by X the real line $\mathbb{R}$ equipped with the discrete metric (taking values $0, 1$ only), and let $Y = \mathbb{R}$ be the real line with the standard metric. Denote by f the identity mapping from $\mathbb{R}$ to itself, $f(x) = x$, viewed as a map from X to Y. This f is surjective and also continuous: if $x \in \mathbb{R}$, then, no matter what $\epsilon > 0$ is, set $\delta = 1$, and one has

$$f(B_1(x)) = f(\{x\}) = \{f(x)\} \subseteq B_\epsilon(f(x)).$$

The set $\mathbb{Q}$ of rationals is everywhere dense in Y, but the inverse image $f^{-1}(\mathbb{Q}) = \mathbb{Q}$ is a closed subset of X (as every other subset), and so

$$\operatorname{cl} f^{-1}(\mathbb{Q}) = \mathbb{Q} \subsetneqq \mathbb{R} = X.$$

Without an assumption that f be surjective, the problem becomes easier: take, for instance, as X any non-empty metric space whatsoever, and let f

map X to $\mathbb{R}$ in a constant fashion: $f(x) = \sqrt{2}$ for all $x \in X$. This mapping is of course continuous. Now take $B = \mathbb{Q}$, the set of rational numbers, which is dense in $\mathbb{R}$. The inverse image $f^{-1}(\mathbb{Q})$ is empty, and so not dense in X.

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(3) (a) **[2 points]** *State the definition of a disconnected metric space; of a connected metric space.*

Definitions 6.1 and 6.4.

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(b) **[2 points]** *Prove that a continuous image of a connected metric space is connected.*

Theorem 7.10.

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(c) **[2 points]** *Is it true or false that a continuous image of a disconnected space is disconnected? If yes, give a proof. If no, construct a counter-example.*

It is false in general. Here is the simplest example possible. Denote by $A = \{0, 1\}$ a two-element space with a metric given by $d(x, y) = 1$. Let $Y = \{*\}$ be a singleton space, with a trivial metric. The function $f\colon X \to Y$ taking every element of X to $*$ is continuous, as follows, for instance, from an argument employed in Problem 2(d) above. The space X is obviously disconnected: $\{0\}$ and $\{1\}$ form non-empty disjoint open subsets (each of them coincides with the open 1-ball around itself). The space Y is connected, because if $Y = U \cup V$ and U is non-empty, then $* \in U$ and $V = \emptyset$.

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(d) **[2 points]** *A metric space X is called* totally disconnected *if every metric subspace Y of X, containing more than one point, is disconnected. Prove that the space $\mathbb{Q}$ of rational numbers with the usual distance is totally disconnected.*

Let X be a metric subspace of $\mathbb{Q}$ containing two distinct points, x and y. Without loss in generality, one can assume that $x < y$. Find an irrational point z with the property $x < z < y$, and set

$$V = \{a \in X\colon a < z\},\ U = \{a \in X\colon a > z\}.$$

Clearly, V and U are disjoint, and since $z \notin X$, the union of V and U gives all of X. Following the same lines as our proof of disconnectedness of $\mathbb{Q}$ in Example 6.9, one can easily show that U and V are both open. Finally, since

$x \in V$ and $y \in U$, both sets are non-empty. We conclude: X is disconnected, and so $\mathbb{Q}$ with the standard metric is a totally disconnected metric space.

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