uOttawa
Department of Mathematics and Statistics
Département de mathématiques et statistique

Analysis III – Mat 3120

Final Exam — December 7, 2006

Time: 3 hours.
Attempt ALL four questions.
Each question is worth 13 marks.
Total number of marks: 52
This is a closed book exam.
No electronic devices are allowed.

(1) (a) Give the definition of an open subset of a metric space. **[1 mark]**

(b) Let X be a metric space, and let Y be a subspace of X. Suppose a set $V \subseteq X$ is open in X. Prove that the set $V \cap Y$ is open in Y. **[2 marks]**

(c) Give the definition of the boundary $\operatorname{bd} A$ of a subset A in a metric space X. **[1 mark]**

(d) Let A be a subset of a metric space (X, d). Prove that the boundary of A is empty if and only if A is an open and closed subset of X. **[4 marks]**

(e) State the definition of a connected metric space. **[1 mark]**

(f) Let X be a metric space and let $x \in X$. The *component* of x in X is the union of all connected subsets of X containing x. It is denoted by $C(x)$. Show that the set $C(x)$ is connected (as a metric subspace of X). **[5 marks]**

[Continued on the next page....]

(2) (a) Define what it means that a mapping $f\colon X \to Y$ between two metric spaces is continuous. **[1 mark]**

(b) Let $X = (X, d)$ be a metric space. Let $F \subseteq X$ be a non-empty closed subset of X. Define a real-valued function d_F on X by letting for each $x \in X$:

$$d_F(x) = \inf\{d(x, y)\colon y \in F\}$$

(the distance from x to F).

(i) Prove that the function d_F assumes non-negative values. **[1 mark]**

(ii) Show that the function d_F is continuous. **[3 marks]**

(iii) Prove that $d_F(x) = 0$ if and only if $x \in F$. **[3 marks]**

(iv) Let $f\colon X \to \mathbb{R}$ be an arbitrary continuous function. Prove that the set

$$\mathcal{Z}_f = \{x \in X\colon f(x) = 0\}$$

is the intersection of inverse images $f^{-1}(B_\epsilon(0))$ under f of open balls in $\mathbb{R}$ around zero of all positive radii $\epsilon > 0$. **[3 marks]**

(v) Recall that a subset A of a metric space X is a G_δ *set* if A is the intersection of a countable family of open sets. Deduce from the above results that in a metric space every closed subset F is a G_δ set. **[2 marks]**

(3) (a) State the definition of an everywhere dense subset of a metric space. **[1 mark]**

(b) Let X be a metric space and Y an everywhere dense subspace of X. Suppose that Y is a connected metric space. Prove that X is connected as well. **[4 marks]**

(c) Give the definition of a compact metric space. **[1 mark]**

(d) Give an example of a metric space X having the properties: (i) X is infinite, and at the same time (ii) every compact subspace of X is finite. Explain. **[3 marks]**

(e) Define what it means that a metric space (X, d) is totally bounded. **[1 mark]**

(f) The First Weierstrass Theorem tells us that every continuous real-valued function on a compact metric space is bounded. Is every continuous real-valued function on a *totally bounded* metric space bounded? If yes, give a proof; if no, construct a counter-example. **[3 marks]**

[Continued on the next page....]

(4) (a) Give the definition of a normed space. **[1 mark]**

(b) Let E be a normed space.

(i) Prove that every element $y \in E$ is a scalar multiple of a suitable element of the open ball $B_1(0)$ of unit radius around zero in E, that is, $y = \lambda x$, where λ is a scalar and $x \in B_1(0)$. **[2 marks]**

(ii) Now let $B_\epsilon(x)$ be an arbitrary open ball in E, where $x \in E$ and $\epsilon > 0$. Prove that the linear span of $B_\epsilon(x)$ in E is all of E, that is, every element $y \in E$ can be written as a finite linear combination of elements of the ball in question. (*Hint:* use the properties of a norm.) **[3 marks]**

(c) Recall that a *straight line*, ℓ, in $\mathbb{R}^2$ can be defined as a set of the form $\ell = \{x \in \mathbb{R}^2 \colon ax_1 + bx_2 = 0\}$, where a, b are some real numbers depending on the line, not both equal to zero. In other words, $\ell = \mathcal{Z}_f$ (cf. problem 2(b)iv), where $f \colon \mathbb{R}^2 \to \mathbb{R}$ is a suitable linear function, $f(x) = ax_1 + bx_2$. Prove that every line is a closed subset of $\mathbb{R}^2$ considered as a normed space $\ell^2(2)$ (that is, equipped with the Euclidean distance). **[3 marks]**

(d) Prove that the interior of every line ℓ in $\mathbb{R}^2 = \ell^2(2)$ is empty. (*Hint:* you may either do it directly or else use problem 4(b)ii.) **[3 marks]**

(e) Prove that the plane $\mathbb{R}^2$ cannot be represented as the union of countably many straight lines. State all the major results from the course that you are using (there will be at least two of them). **[3 marks]**

[End of the exam questions]