Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2141 – The final exam **Instructor: K. Zaynullin**

Last name: ______________________

First name: ______________________

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Question	1	2	3	4	5	6	7	
Mark								
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Question	8	9	10	11	12	13	14	Total
Mark								
Out of	3	3	3	4	3	3	3	50

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1. Label the following statements as true or false. (5)

1. Let $f: V \to W$ be a function between two vector spaces over a field F. If $f(x-y) = f(x) - f(y)$ and $f(cx) = cf(x)$ for all $x, y \in V$ and $c \in F$, then f is a linear transformation.
2. Let $A \in M_n(F)$ be an $n \times n$ matrix with entries in a field F. If $A^2 = 0$ (the zero matrix), then $A = 0$.
3. The vector space $M_3(\mathbb{C})$ is isomorphic to $\mathbb{C}^3$.
4. Let V and W be finite-dimensional vector spaces. Let $\mathcal{L}(V, W)$ denote the vector space of all linear maps from V to W. Then $\mathcal{L}(V, W)$ is always finite-dimensional.
5. Every transformation matrix is invertible.

2. Label the following statements as true or false. (5)

1. Two similar matrices have always the same rank.
2. The inverse of an elementary matrix is the same matrix.
3. A homogeneous system of linear equations has always a solution.
4. If columns of a square matrix A are linearly dependent, then $\det(A) = 0$.
5. The determinant function $\det\colon M_2(\mathbb{C}) \to \mathbb{C}$ is always surjective.

3. Label the following statements as true or false. (5)

1. A $n \times n$-matrix is diagonalizable if it has n distinct eigenvectors.
2. The dimension of an eigenspace of a linear operator T can be equal to the dimension of the image of T.
3. If $\langle x, x \rangle > 0$ in an inner product space, then $||x|| > 0$.
4. Any subspace of a finite-dimensional inner product space has an orthonormal basis.
5. The orthogonal complement of any linear subspace is again a subspace.

4. Label the following statements as true or false. (5)

1. Any linear operator T on an inner product space has an adjoint T^*.
2. If T is normal, then $-T$ is normal.
3. A composite of isometry operators is always normal.
4. A rigid motion is always a composite of isometry operators.
5. Eigenvalues of a self-adjoint operator on a finite-dimensional vector space are real numbers.

5. Determine whether the following set is a subspace of $\mathbb{C}^2$ under the operations of addition and scalar multiplication defined on $\mathbb{C}^2$. Justify your answer. (3)

$$W = \{(z_1, z_2) \in \mathbb{C}^2 \mid (z_1 + iz_2)^2 = 0\}$$

6. Determine whether the following function $T\colon \mathbb{C}^2 \to \mathbb{C}$ is a linear transformation

$$T(z_1, z_2) = |z_1| + |z_2|.$$

Justify your answer. (3)

7. Let V and W be finite-dimensional vector spaces and $T\colon V \to W$ be a linear map. Prove that if $\dim(V) > \dim(W)$, then T cannot be injective. (2)

8. Let $T\colon \mathbb{R}[t]_2 \to \mathbb{R}[t]_2$ be a linear transformation defined by

$$T(p(x)) = p''(x) - p'(x).$$

Determine whether T is invertible. Justify your answer. (3)

(here $\mathbb{R}[t]_2$ is the vector space of polynomials of degree at most 2)

9. Let $\delta\colon M_3(F) \to F$ be a function defined by $\delta(A) = a_{23}$ (the entry of $A = (a_{ij})$ at the position $(2, 3)$). Determine, whether δ is a multilinear function or not. Justify your answer. (3)

10. Suppose that a matrix $A \in M_n(F)$, $n \geq 2$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n - 1$. Prove that A is diagonalizable. (3)

11. Let $V = C([0,\pi])$ be the vector space of all real-valued continuous functions on the interval $[0,\pi]$ with the inner product defined by

$$\langle f, g\rangle = \int_0^{\pi} f(t)g(t)\,dt.$$

(a) Show that the subset $S = \{\sin t, \cos t\}$ of V is linearly independent. (2)

(b) Find an orthonormal basis of $Span(S)$ applying the Gram-Schmidt process to S. (2)

12. Let $\mathbb{R}[t]$ be the vector space of polynomials with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt.$$

Find the orthogonal projection of the polynomial $t^2 + 1$ on the subspace $\mathbb{R}[t]_1$ of polynomials of degree at most 1. (3)

13. Give an example of a linear operator T on an inner product space V such that $im(T) \neq im(T^*)$. (3)

14. For $z \in \mathbb{C}$, define a linear operator $T_z \colon \mathbb{C} \to \mathbb{C}$ by $T_z(u) = zu$. Describe all z such that T_z is a unitary operator. (3)

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