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MAT 2141: Final Exam ANSWERS

December 22, 2015 Prof. Philip Scott

In what follows, $V, W, \ldots$ denote vector spaces over some field $\mathbb{K}$.

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1. (a) Let V be a vector space. Define what it means to say: "A subset $S = \{x_1, \cdots, x_n\} \subseteq V$ is linearly dependent".

   (b) Let $T \in Lin(V, W)$ be a linear map and $\{x_1, \cdots, x_n\} \subseteq V$. Prove: if $\{T(x_1), \cdots, T(x_n)\}$ is linearly independent, then so is $\{x_1, \cdots, x_n\}$.

Ans: (a) $\exists$ scalars $\alpha_1, \ldots, \alpha_n$ not all $= 0$, such that $\sum_{i=1}^{n} \alpha_i x_i = \theta$.

(b) Suppose $\{T(x_1), \ldots, T(x_n)\}$ is linearly independent.
Suppose $\sum_{i=1}^{n} \alpha_i x_i = \theta$. $\therefore T(\sum_{i=1}^{n} \alpha_i x_i) = T\theta = \theta$. Hence
$\sum_{i=1}^{n} \alpha_i T(x_i) = T(\sum_{i=1}^{n} \alpha_i x_i) = T\theta = \theta$. Hence by indep. of
$\{Tx_1, \cdots, Tx_n\}$, all the $\alpha_i = 0$ ✓.

2. (a) Define: U is the subspace of V generated (or spanned) by the set $\{x_1, \cdots, x_n\}$.

   (b) Prove: If $\{x_1, x_2, x_3\}$ generate U, then $\{x_1, x_1 + x_2, x_1 + x_2 + x_3\}$ also generates U.

(a) $U = L(\{x_1, \ldots, x_n\}) = \{ y \mid y = \sum_{i=1}^{n} \alpha_i x_i$, for some $\alpha_1, \ldots, \alpha_n \in \mathbb{K} \}$.

(b) Prove $L(x_1, x_2, x_3) = L(x_1, x_1 + x_2, x_1 + x_2 + x_3)$. Method 1 : Prove 2 inclusions.
($\supseteq$) Obvious, since any linear comb. of RHS is one of LHS. ✓
($\subseteq$) Suppose $U = L(x_1 x_2 x_3)$. Then $\forall y \in U$, $y = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, for some $\alpha_i \in \mathbb{K}$.
To show $y \in L(\{x_1, x_1 + x_2, x_1 + x_2 + x_3\})$, we must find $\beta_1, \beta_2, \beta_3$ s.t.
$\beta_1 x_1 + \beta_2 (x_1 + x_2) + \beta_3 (x_1 + x_2 + x_3) = y = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$.
$\therefore$ $(\beta_1 + \beta_2 + \beta_3) = \alpha_1$
$(\beta_2 + \beta_3) = \alpha_2$
$\beta_3 = \alpha_3$
} equating coefficients of x_1, x_2, x_3.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & \alpha_1 \\ 0 & 1 & 1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \alpha_1 - \alpha_2 \\ 0 & 1 & 1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_3 \end{array}\right] \to \left[\begin{array}{c|c} I & \begin{matrix} \alpha_1 - \alpha_2 \\ \alpha_2 - \alpha_3 \\ \alpha_3 \end{matrix} \end{array}\right]$$

$\therefore$ $\beta_1 = \alpha_1 - \alpha_2$, $\beta_2 = \alpha_2 - \alpha_3$, $\beta_3 = \alpha_3$

This proves the other inclusion. Method 2 is a bit easier:

1(b) Method 2: Suppose S, S' are two sets of vectors. To prove $L(S) = L(S')$ it suffices to prove:
(i) $S \subseteq L(S')$ and (ii) $S' \subseteq L(S)$
This is because $L(S)$ is the smallest subspace that contains S. Hence if (i) is true, $L(S) \subseteq L(S')$ and dually if (ii) is true, $L(S') \subseteq L(S)$ $\therefore$ if (i) and (ii) are true, $L(S) = L(S')$.
Hence to prove $L(x_1, x_2, x_3) = L(x_1, x_1+x_2, x_1+x_2+x_3)$ it suffices to prove
(a) $\{x_1, x_2, x_3\} \subseteq L(x_1, x_1+x_2, x_1+x_2+x_3)$ and
(b) $\{x_1, x_1+x_2, x_1+x_2+x_3\} \subseteq L(x_1, x_2, x_3)$.

Re (a), $x_1 = 1 \cdot x_1$, $x_2 = -1 \cdot x_1 + 1 \cdot (x_1+x_2)$, $x_3 = -1(x_1+x_2) + 1 \cdot (x_1+x_2+x_3)$

$\therefore x_1, x_2, x_3 \in L(x_1, x_1+x_2, x_1+x_2+x_3)$ as desired.

Re (b), This is obvious — each of x_1, x_1+x_2, $x_1+x_2+x_3$ is clearly a linear combination of x_1, x_2, x_3. ✓

3. (a) Define the notion "$\mathcal{B}$ is a basis of the $\mathbb{K}$-vector space V".

   (b) Suppose $\{v_1, \ldots, v_n\}$ is a basis of V. Set $M_0 = \{\theta\}$, and for each $k \in \mathbb{Z}^+$, set $M_k = L(\{v_1, \ldots, v_k\})$, the linear span of $\{v_1, \ldots, v_k\}$. For each $k \in \{1, \cdots, n\}$, pick $y_k \in M_k$ such that $y_k \notin M_{k-1}$. This is always possible (why?). Prove that $\{y_1, \ldots, y_n\}$ is also a basis of V.

(a) $\mathcal{B}$ is a linearly independent generating set.

(b) For $k \geq 1$, $M_k = L(v_1, \ldots, v_k)$. $\exists\, y_k \in M_k \setminus M_{k-1}$, since if $M_k = M_{k-1}$, then $L(v_1 \cdots v_k) = L(v_1 \cdots v_{k-1})$. In particular, $v_k \in M_{k-1}$. Impossible since $v_1, \ldots, v_{k-1}, v_k$ are linearly independent.

$\{y_1 \cdots y_n\}$ is independent: Suppose $y_1 \cdots y_n$ is dependent. Then $\exists k$ s.t. $y_k \in L(y_1, \ldots, y_{k-1})$. But $L(y_1, \ldots, y_{k-1}) \subseteq M_{k-1}$, since $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{k-1}$.
$\therefore y_k \in M_{k-1}$. Contradiction.
$\therefore$ since $\dim(V) = n$, $\{y_1, \ldots, y_n\}$ automatically spans (since it's independent.)
$\therefore \{y_1, \ldots, y_n\}$ is a basis. ✓

4. Let $T \in Lin(V, W)$ be a linear map between two $\mathbb{K}$-vector spaces.

   (a) Define $Ker(T)$.

   (b) Let $w \in W$ be a vector. Suppose there exists a $v_0 \in V$ such that $T(v_0) = w$. Prove that for any $v \in V$, $T(v) = w$ iff $v = v_0 + u$, for some $u \in Ker(T)$.

Ans: (a) $\ker(T) = \{v \mid Tv = \theta\}$.

(b). Suppose $\exists v_0 \in V$ s.t. $Tv_0 = w$.

($\Rightarrow$) Suppose $Tv = w$. $\therefore Tv = w = Tv_0$ $\therefore T(v - v_0) = \theta$.
$\therefore v - v_0 \in \ker(T)$. $\therefore v = v_0 + (v - v_0)$. Let $u = v - v_0 \in \ker(T)$ ✓.

($\Leftarrow$) Suppose $v = v_0 + u$ for some $u \in \ker(T)$.

$\therefore Tv = Tv_0 + Tu = Tv_0 = w$, since $Tu = \theta$.

5. Let $T \in Lin(V, V)$, where V is a finitely generated vector space.

   (a) Define the notions "eigenvalue" and "eigenvector" of T.

   (b) Let $v \in V$ be a nonzero vector. Let $W = L(\{v\})$, the span of v. Prove: v is an eigenvector of T iff $T(W) \subseteq W$.

   In the terminology of our class, this says "v is an eigenvector of T iff the space W is T-invariant."

(a) $\lambda \in \mathbb{K}$ is an eigenvalue of T if $\exists\, v \in V, v \neq \theta$, s.t. $Tv = \lambda v$.
$v \in V$ is an eigenvector of eigenvalue λ if $Tv = \lambda v$ (this allows the possibility $v = \theta$!)

(b) Prove: $v \in$ Eigenvectn T iff W is T-invariant.

($\Rightarrow$) Suppose $Tv = \lambda v$ for some λ. We must prove $TW \subseteq W$. Let $Tw \in TW$. $\therefore$ for some $\mu \in \mathbb{K}$, $w = \mu v \in W$. $\therefore Tw = T(\mu v) = \mu Tv = \mu\lambda v = \lambda \mu v = \lambda w \in W$. $\therefore TW \subseteq W$ ✓

($\Leftarrow$) Suppose $T(W) \subseteq W$. $\therefore$ Since $v \in W, v \neq \theta$, $Tv \in W$. $\therefore \exists\, \lambda \in \mathbb{K}$ s.t. $Tv = \lambda v$. $\therefore v$ is an eigenvector of T (assoc. to e.v. λ).

6. Let $U, W \subseteq V$ be two subspaces of a finite dimensional vector space over $\mathbb{K}$. Consider the following linear map:

$$f : U \times W \to U + W$$
$$(u, w) \mapsto u - w$$

   (a) Prove that f is surjective.

   (b) Describe (up to isomorphism) the subspace $Ker(f)$.

   (c) Using Rank-Nullity, prove the following formula of Grassmann:

$$dim(U) + dim(W) = dim(U \cap W) + dim(U + W)$$

   (d) If $dim(V) = 10, dim(U) = 4$ and $dim(W) = 7$, then prove that there exists a vector $v \neq \theta$ such that $v \in U \cap W$.

(a) Let $y \in U + W$. $\therefore \exists\, u \in U, w' \in W$ s.t. $y = u + w'$.
Then $f(u, -w') = u - (-w') = u + w' = y$. $\therefore f$ is surjective.

(b) $Ker(f) = \{(u, w) \in U \times W \mid f(u, w) = \theta\}$
$= \{(u, w) \in U \times W \mid u = w\} \cong \{v \in V \mid v \in U \cap W\} = U \cap W$
(Here the isos are: $(u, w) \longmapsto u$, $(v, v) \longleftarrow\!\shortmid v$ clearly linear and bijective).

(c) By R-N, $\dim(U \times W) = \dim(Im(f)) + \dim(Ker(f))$. Hence (by (a)) $\dim U + \dim W = \dim(U + W) + \dim(U \cap W)$ ✓

(d) $U \underset{\text{space}}{\subseteq} U + W \underset{\text{space}}{\subseteq} V$ and $W \underset{\text{space}}{\subseteq} U + W \underset{\text{space}}{\subseteq} V$. $\therefore \dim U + \dim W = 11$
But $\dim(U + V) \leq \dim(V) = 10$. Hence $\dim(U \cap V) = \dim(Ker(f)) \geq 1$ ✓

7. Let V be a vector space over $\mathbb{K}$ of dimension n and let $T \in Lin(V,V)$ be a linear map such that $dim(Im(T)) = 1$.

   (a) Define: T is a diagonalisable linear operator.

   (b) Prove that T is diagonalisable.

   (c) Prove that there exists a scalar $\lambda \in \mathbb{K}$ such that $T^2 = \lambda T$. (Hint: start with a basis of $Im(T)$ then extend appropriately to all of V).

(a) For any basis $\mathcal{B}$ of V, $[T]_{\mathcal{B}}^{\mathcal{B}} \sim D$, D diagonal.

(b) By Rank-Nullity, $Dim(V) = dim(Im(T)) + dim(Ker(T))$.

Suppose $\{v_1\}$ = basis of $Im(T)$. $\therefore Dim(Ker(T)) = n-1$.

Let $\{v_2, \dots, v_n\}$ be a basis of $Ker(T)$. Then I claim $v_1 \notin Ker(T)$. Since if $v_1 \in Ker(T) \Rightarrow Tv_1 = \Theta \Rightarrow Im(T) = T(L(v_1)) = \{\Theta\}$
$\Rightarrow dim(Im(T)) = 0$. ✗

$\therefore \mathcal{B} = \{v_1, v_2, \dots, v_n\}$ is a basis of V. Notice: $Tv_1 = \lambda v_1$ for some $\lambda \in \mathbb{K}$; $T(v_i) = \Theta = 0\, v_i$ for $i \geq 2$.

Hence $\{\lambda, 0\}$ are the eigenvalues of T, and for the eigenvectors $v_1, \dots, v_n$, $[T]_{\mathcal{B}}^{\mathcal{B}} \sim \begin{bmatrix} \lambda & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$ w.r.t. the similarity matrix $P = [v_1 | v_2 | \cdots | v_n]$.

(c) $Tv_1 = \lambda v_1$. $T(v_2) = \cdots = T(v_n) = \Theta$. (*)

If $v = \sum_{i=1}^{n} \alpha_i v_i$, $T^2 v = T(Tv) = T\left(\sum_i \alpha_i Tv_i\right) = T(\alpha_1 Tv_1)$ by (*).

$= T(\alpha_1 \lambda v_1) = \lambda(T(\alpha_1 v_1)) = \lambda[T(\alpha_1 v_1 + \cdots + \alpha_n v_n)]$ } since T linear & $Tv_i = \Theta$

$= \lambda T(v)$.

$\therefore T^2 v = \lambda T v$, $\forall v \in V$

$\therefore T^2 = \lambda T \in Lin(V, V)$.

8. Let V be a finite dimensional vector space over $\mathbb{C}$, the complex numbers.

   (a) Define an Hermitian inner product.

   (b) Consider the following two matrices:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Classify (with an explanation) if these are Hermitian or Unitary.

   (c) In the case of Hermitian and Unitary operators, the eigenvalues must satisfy precise conditions (in each case). What are these conditions?

   (d) Calculate the eigenvalues of A and B and verify they satisfy the conditions in part (c) above.

   (e) Find an orthonormal basis of eigenvectors associated to the Hermitian matrix in (b) above.

Ans: (a) $\langle - | - \rangle : V \times V \to \mathbb{C}$ s.t.

(i) $\langle v|v\rangle \geq 0$ and $\langle v|v\rangle = 0$ iff $v = \theta$

(ii) $\langle u+w|z\rangle = \langle v|z\rangle + \langle w|z\rangle$, $\langle \alpha v|w\rangle = \alpha\langle v,w\rangle$, $\alpha \in \mathbb{C}$.

(iii) $\langle v|w\rangle = \overline{\langle w|v\rangle}$

(b) A, B are real matrices. Since $A^t = A$, A is Hermitian.

Consider $B^t = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $B^tB = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$.

$\therefore B^t = B^{-1}$, so B is unitary.

(c) Hermitian: eigenvalues are real.

Unitary: for all eigenvalues λ, $|\lambda| = 1$.

(d) $|\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & -2 \\ 0 & \lambda+1 & 0 \\ -2 & 0 & \lambda-1 \end{vmatrix} = (\lambda+1)\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda+1)\left[(\lambda-1)^2 - 4\right]$

$= (\lambda+1)(\lambda^2 - 2\lambda + 1 - 4) = (\lambda+1)(\lambda^2 - 2\lambda - 3) = (\lambda+1)^2(\lambda-3)$.

Hence $\lambda = -1$ (repeated), $\lambda = 3$. Note $\lambda \in \mathbb{R}$.

$|\lambda I - B| = \begin{vmatrix} \lambda & -1 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = (\lambda-1)\left[\lambda^2 - 1\right]$ $\therefore \lambda = 1, \pm i$

Notice $|\lambda| = 1$.

(e) For A, Eigensp$(-1) = \ker\begin{pmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & -2 \end{pmatrix} = \ker\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{matrix} x_1 = -t \\ x_2 = s \\ x_3 = t \end{matrix}$

$\therefore \vec{x} = s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $s,t \in \mathbb{R}$. $\therefore$ Eigensp$(-1) = L\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Normalizing: Eigensp$(-1) = L\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}\right\}$.

$$\text{Eigensp}(3) = \ker\begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix} = \ker\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \begin{matrix} x_1 = s \\ x_2 = 0 \\ x_3 = s \end{matrix} \quad \therefore \vec{x} = s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$\therefore \text{Eigensp}(3) = L\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

Let $P = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Its columns are an o.n. basis of eigenvectrs.

Note: $P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Choose two of the three following questions (extra credit: try them all)

9. Let V be a finite dimensional vector space over $\mathbb{K}$ and let $T \in Lin(V,V)$ be such that $T^2 = T$.

   (a) Prove T is diagonalizable. What are the possible eigenvalues of T?

   (b) Suppose V has dimension 3. Let $\mathcal{B}$ be a basis of 3 eigenvectors. Consider the matrix $[T]^{\mathcal{B}}_{\mathcal{B}}$. Up to permutation of the eigenvectors, what are the possible forms of $[T]^{\mathcal{B}}_{\mathcal{B}}$?

Ans: Recall from class, we can write
$\forall v \in V$, $v = (v - Tv) + Tv$. Also, if $T^2 = T$,
notice $T(v - Tv) = Tv - T^2v = Tv - Tv = \theta$.
$\therefore v - Tv \in \ker(T)$ and $Tv \in \mathrm{Im}(T)$
$\therefore$ We can write $V = \ker(T) \oplus \mathrm{Im}(T)$
The sum is a direct sum, since if
$Tv \in \ker(T) \cap \mathrm{Im}(T)$ then $T(Tv) = \theta$. But $T^2 = T$. $\therefore Tv = \theta$. ✓
$\therefore \ker(T) \cap \mathrm{Im}(T) = \{\theta\}$.

As mentioned in class,
$\ker(T) = \{v \in V \mid Tv = \theta\} = \{v \in V \mid Tv = 0 \cdot v\} = \text{Eigensp}(0)$.
Similarly
$\mathrm{Im}(T) = \{Tv \mid v \in V\} = \{v \mid Tv = v\} = \text{Eigensp}(1)$.
Hence the eigenvalues of T are $\lambda = 0, 1$ and V has a basis consisting of linearly independent eigenvectors (by merging the bases of $\ker(T)$ and $\mathrm{Im}(T)$).
$[T]^{\mathcal{B}}_{\mathcal{B}}$ is any diagonal matrix with 1's and 0's on the main diagonal: e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (up to permutation)
together with the extreme cases $I_{3\times 3}$ and $O_{3\times 3}$. For these,
for $I_{3\times 3}$, $\lambda = 1$ and $\text{Eigensp}(1) = V$, since $Iv = v$ $\forall v \in V$.
for $O_{3\times 3}$, $\lambda = 0$ and $\text{Eigensp}(0) = V$, since $Ov = \vec{0} = 0v$, $\forall v \in V$.

10. Let T be a unitary operator on a unitary space U. Suppose λ is an eigenvalue of T.

    (a) Prove the following Lemma (which was an exercise in our notes):

    $$\text{If } Tv = \lambda v \text{ then } T^*v = \bar{\lambda}v.$$

    [Hint: Calculate $J(v)(T^*v)$. You may assume the map $J : U \to U^*$, where $J(v) = \langle -|v\rangle$ is conjugate linear and has the same properties as in class.]

    (b) Using the previous part (a) for a unitary operator T, calculate $|\lambda|$, for any eigenvalue λ.

(I will not count this, except as extra extra credit).

Proof: (a) $\|Tx\| = \|T^*x\|$, since $J(x)(T^*Tx) = \langle T^*Tx|x\rangle = \langle Tx|Tx\rangle = \|Tx\|^2$

But $T^*T = TT^*$. $\therefore$ $\langle TT^*x|x\rangle = \langle T^*x|T^*x\rangle = \|T^*x\|^2$ and

$\langle TT^*x|x\rangle = \langle T^*Tx|x\rangle$. $\therefore$ $\|Tx\| = \|T^*x\|$.

if $Tv = \lambda v$, $\|(T-\lambda I)v\| = 0$. $\therefore$ $\|(T-\lambda I)^*v\| = 0$. $\therefore$ $T^*v = \bar{\lambda}v$.

(b) Here's a direct proof:

We know $T^* = T^{-1}$. Let v be an eigenvector of T, so $v \neq 0$.

$\therefore$ $\langle v|v\rangle = \|v\|^2 \neq 0$. Hence

$\bar{\lambda}\lambda\langle v|v\rangle = \bar{\lambda}\langle \lambda v|v\rangle = \langle \lambda v|\lambda v\rangle = \langle Tv|Tv\rangle = \langle v|T^*Tv\rangle = \langle v|v\rangle$

since $T^*T = I$ (T is unitary)

Since $\langle v|v\rangle = \|v\|^2 \neq 0$, $\bar{\lambda}\lambda = 1$ $\therefore$ $|\lambda| = 1$ ✓

11. (a) Consider $\mathbb{C}$ with its usual Hermitian dot product. Consider the linear operator $T : \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$T(x, y, z) = (ix + (2 + 3i)y, 3x + (3 - i)z, (2 - 5i)y + iz)$$

Calculate $T^*(x, y, z)$ (Hint: find the matrix of T in the standard basis).

(b) Let T be a linear map between inner product spaces over $\mathbb{C}$. Prove $(cT)^* = \bar{c}T^*$, for $c \in \mathbb{C}$. Which properties of the adjoint do you use?

(c) Let T be a self-adjoint operator on a Unitary space V. Prove that for all $x \in V$,

$$||T(x) + ix||^2 = ||T(x)||^2 + ||x||^2$$

Be precise on what properties of the adjoint and inner products you use.

(a) $T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}i\\3\\0\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2+3i\\0\\2-5i\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\3-i\\i\end{pmatrix}$

$\therefore [T] = \begin{bmatrix} i & 2+3i & 0 \\ 3 & 0 & 3-i \\ 0 & 2-5i & i \end{bmatrix}$ $\therefore T^* = \overline{T}^t = \begin{bmatrix} -i & 3 & 0 \\ 2-3i & 0 & 2+5i \\ 0 & 3+i & -i \end{bmatrix}$

$\therefore [T^*]\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix} -ix + 3y \\ (2-3i)x + (2+5i)z \\ (3+i)y - iz \end{pmatrix}$

(b) $\langle (cT)u | v \rangle = \langle cT(u) | v \rangle = c\langle T(u) | v \rangle = c\langle u | T^* v \rangle = \langle u | \bar{c} T^* v \rangle$
$= \langle u | (\bar{c} T^*) v \rangle$

But we also know $\langle (cT)u | v \rangle = \langle u | (cT)^* v \rangle$. So by uniqueness of the adjoint, $(cT)^* = \bar{c} T^*$.

In addition to uniqueness, this also used the fact that $\langle | \rangle$ is conjugate linear in the second argument, linear in the first argument, and satisfies the defining equation $\langle T(\cdot) | - \rangle = \langle - | T^* - \rangle$.

(c) $\langle Tx + ix | Tx + ix \rangle = \langle Tx | Tx + ix \rangle + \langle ix | Tx + ix \rangle$

$= \langle Tx | Tx \rangle + \langle Tx | ix \rangle + \langle ix | Tx \rangle + \langle ix | ix \rangle$

$= ||Tx||^2 + -i\langle Tx | x \rangle + i\langle x | Tx \rangle + i\bar{i}\langle x | x \rangle$

$= ||Tx||^2 - i\langle Tx | x \rangle + i\langle Tx | x \rangle + ||x||^2$

(since T is self-adjoint and $i\bar{i} = +1$)

$= ||Tx||^2 + ||x||^2$.