Family Name (Capital Letters) _____

First Name (Capital Letters) _____

Signature _____

Student Number _____

Mat 2141 Test 1

Professor: P. Scott Date: October 9, 2014 Time: 75 minutes.

Instructions: No calculators, no books, no notes allowed.

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1. Consider the linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^4$, defined by a matrix A, where $T_A(\vec{x}) = A\vec{x}$.

(i) What is the size of the matrix A?

(ii) Can T_A be injective? If so, what are the row reduced echelon forms of the matrices A for which T_A is injective?

(iii) Can T_A be surjective? Explain, with an example of such an A if it is possible, and otherwise a proof that it's impossible.

2.

Suppose U, V, W are vector spaces and $S : U \to V$ and $T : V \to W$ are linear maps. Show:

If S is injective and $Im(S) \cap Ker(T) = \{\underline{0}\}$ then TS is injective (where TS means $T \circ S$).

3. For each of the sentences below, write **T** or **F** in the following grid :

a	b	c	d	e	f	g	h

(a) For any field $\mathbb{K}$ and elements $c, x_1, x_2 \in \mathbb{K}$, if $cx_1 = cx_2$ then $x_1 = x_2$.

(b) For any field $\mathbb{K}$ and vector space V over $\mathbb{K}$, if $c \in \mathbb{K}$ and $v \in V$ satisfy $cv = \underline{0}_V$ then $c = 0$ and $v = \underline{0}_V$.

(c) There exists a vector space V and subspaces M, N of V such that $M \cup N$ is a subspace of V.

(d) There exists a vector space V and subspaces M, N of V such that $M \cup N$ is not a subspace of V.

(e) There exists a vector space V with subspaces M, N of V such that $M \cap N$ is not a subspace of V.

(f) $\{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\}$ is a subset of $\{(x, y, z) \in \mathbb{R}^3 \mid xy = 0\}$.

(g) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (yz, xz, xy)$ is a linear map.

(h) For any sets X, Y and any function $g : X \to Y$, the mapping

$$T_g : Func(Y, \mathbb{R}) \to Func(X, \mathbb{R}), \quad \text{where} \quad T_g(f) = f \circ g \text{ for all } f \in Func(Y, \mathbb{R})$$

is a linear mapping.

4. Let $\mathbb{S}$ be the real vector space whose elements are infinite sequences $(x_0, x_1, x_2, \dots)$ such that $x_i \in \mathbb{R}$ for all i. Given $x = (x_0, x_1, x_2, \dots) \in \mathbb{S}$, we define

$$\text{supp}(x) = \{\, i \in \mathbb{N} \mid x_i \neq 0\} \ ;$$

the subset $\text{supp}(x)$ of $\mathbb{N}$ is called the *support* of the sequence x. We define the set

$$U = \{\, x \in \mathbb{S} \mid \text{supp}(x) \text{ is a finite set}\}$$

(recall: a set is finite if it has a finite number n of elements, where $n \geq 0$.).

(a) Give an example of an element of U.
(b) Give an example of an element of $\mathbb{S} \setminus U$.
(c) Prove that U is a subspace of $\mathbb{S}$.

5. Let $X = \{1, 2, 3\}$. Let $W = \{f \in Func(X, \mathbb{R}) \mid f(1) = f(3)\}$.

Prove:

(i) W is a subspace of $Func(X, \mathbb{R})$.

(ii) $W \cong \mathbb{R}^2$ (that is, you must construct, with a proof, a linear isomorphism between these two spaces).

6. Let $T : V \to V$ be a linear map between vector spaces. Prove:

$$T^2 = \underline{0} \text{ (the zero map)} \ \Leftrightarrow Im(T) \subseteq Ker(T).$$