# Mat 2141 Final Exam

Professor : Philip Scott Date : December 10, 2014 Time: 3 hours

Family Name (CAPITAL LETTERS) ANSWERS

First Name (CAPITAL LETTERS) _____

Signature _____

Student Number _____

Space reserved for professor. Rough marking scheme shown, but it may change.

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**Questions (Give complete and detailed answers)**

(1) (a) Give a basis for the kernel of $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$.

$\ker(T) = \{A \mid TA = O_{2\times 2}\} = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \middle| \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right\} = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \middle| a = b = 0\right\} = \left\{\left(\begin{smallmatrix} 0 & 0 \\ c & d \end{smallmatrix}\right) \middle| c, d \in \mathbb{R}\right\}$

$= \operatorname{span}\left\{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right\}$ ← this set is clearly independant ∴ a basis.

(b) Give a basis for the image of $T : M_2(\mathbb{R}) \to \mathbb{R}^3$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (a, b, 0)$.

$\operatorname{Im}(T) = \left\{T\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \middle| a, b, c, d \in \mathbb{R}\right\} = \{(a, b, 0) \mid a, b \in \mathbb{R}\} = \operatorname{span}\{e_1, e_2\}$

← basis

(2) Let $V = \mathbb{R}^3$, $W = \mathbb{R}^2$; let $T : V \to W$ be the linear mapping defined by $T(x, y, z) = (x, z)$, and $T' : W^* \to V^*$ the dual map (adjoint) of $T$ (using Berberian's notation). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x - y$ (then $f \in W^*$). Let $g = T'(f)$. Calculate $g(1, 2, 3)$.

$g = T'(f) = f \circ T$. Therefore

$g(1,2,3) = (f \circ T)(1,2,3) = f\big(T(1,2,3)\big) = f(1,3) = 1 - 3 = -2$.

**This page is for extra work**

(3) For each of the following sentences, answer true or false ( **T** or **F** ) and *explain your answer, as defined below.*:

- If you answer '**T**', explain how you know it is true;
- If you answer '**F**', give a concrete counterexample (with numbers!), which shows why the result is false.

(a) If $m \geq 2$ and $T : K^n \to K^m$ is a linear mapping of rank $m$, then $T$ is injective.

False (in general). Let $T: \mathbb{R}^3 \to \mathbb{R}^2 \therefore T = T_A$, $A_{2\times 3}$ a $2\times 3$ matrix. If rank$(A)=2$, Nullity$(A)=1 \therefore \ker(T) \neq \{0\} \therefore T$ is not injective. If $n<m$, also false: rank$(T)$ cannot $=m$ since more rows than cols! If $m=n$ it's always true: if $A_{m\times m}$ has rank $m$, $A_R = Id. \therefore A^{-1}$ exists.

(b) If $V = \mathbb{R}^2$, $W = \mathbb{R}^3$ and $T : V \to W$ is a linear map of rank 2, then the dual map $T' : W^* \to V^*$ satisfies $\dim(\ker T') = 1$.

True $T = T_A$, where $A$ is $3\times 2 . \therefore T' = T_{A^t}$, $A^t$ is $2\times 3$. Hence

Rank$(A)$ = Rank$(A^t) = 2 . \therefore$ Nullity$(T')$ = Nullity$(A^t) = 1$

Since $(A^t)_R \leftarrow$ RREF has shape $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}_{2\times 3}$ (rank 2).

(4) (a) Let $V$ be a real vector space. Define what is ~~a scalar~~ an inner product on $V$.

A function $\langle\,|\,\rangle : V^2 \to \mathbb{R}$ satisfying

① $\langle x|x\rangle \geq 0$ fn all $x \in V$; ③ $\langle x+y|z\rangle = \langle x|z\rangle + \langle y|z\rangle$

② $\langle x|x\rangle = 0$ iff $x = 0$ ④ $\langle \alpha x|y\rangle = \alpha\langle x|y\rangle$, $\forall \alpha \in \mathbb{R}$

⑤ $\langle x|y\rangle = \langle y|x\rangle \quad \forall x, y$

(b) Let $T : V \to W$ be a linear mapping between Euclidean spaces.
(i) Define the canonical map: $J_V : V \to V^*$. and prove it's injective.
(ii) Define the adjoint $T^* : W \to V$. What is the diagram defining the adjoint $T^*$?
(iii) What is the fundamental equation satisfied by the adjoint $T^*$? Is the adjoint of $T$ unique?
(iv) Prove: in a Euclidean space $T^{**} = T$.
(v) Extra Credit. If $V$ has an orthonormal basis $B$, what is the action of $J_V$ on this basis; in particular, what is $D = J_V(B)$? Is it a basis? Compute the matrix $[J_V]^B_D$.

(Write the answer to 4(b) on the next page, or the back.)

(i) $J_V : V \to V^* : y \longmapsto \langle -|y\rangle$.

This is injective: If $J_V(y) = J_V(y') \therefore \langle -|y\rangle = \langle -|y'\rangle$.

$\therefore \forall x, \langle x|y\rangle = \langle x|y'\rangle . \therefore$ from a DGD problem, $y = y'$.

cont'd

4(a) An inner product on a vector space $V$ (over $\mathbb{R}$) is

A function $\langle\,|\,\rangle : V^2 \to \mathbb{R}$ satisfying

① $\langle x|x\rangle \geq 0$ for all $x \in V$; ③ $\langle x+y|z\rangle = \langle x|z\rangle + \langle y|z\rangle$

② $\langle x|x\rangle = 0$ iff $x = \underline{0}$ ④ $\langle \alpha x|y\rangle = \alpha\langle x|y\rangle$, $\forall \alpha \in \mathbb{R}$

⑤ $\langle x|y\rangle = \langle y|x\rangle \quad \forall x, y$

(i) $J_V : V \to V^* : y \longmapsto \langle -|y\rangle$.

This is injective: If $J_V(y) = J_V(y')$ $\therefore \langle -|y\rangle = \langle -|y'\rangle$.

$\therefore \forall x,\ \langle x|y\rangle = \langle x|y'\rangle$. $\therefore$ from a DGD problem, $y = y'$.

(ii)
$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle J_V^{-1}}\uparrow\downarrow{\scriptstyle J_V} & & \downarrow{\scriptstyle J_W} \\ V^* & \xleftarrow{T'} & W^* \end{array} \qquad T^* = J_V^{-1} \cdot T' \cdot J_W$$

(iii) The adjoint satisfies: $\forall x \forall y \left[\langle Tx|y\rangle = \langle x|T^*y\rangle\right]$.

As mentioned in class (and Berberian), $T^*$ is unique:

Any other $S$ satisfying $\langle Tx|y\rangle = \langle x|Sy\rangle$ must satisfy $S = T^*$.

Let's use the symmetric form of $T^*$:

(iv) $\langle T^{**}x|y\rangle = \langle x|T^*y\rangle = \langle T^*y|x\rangle = \langle y|Tx\rangle = \langle Tx|y\rangle$

$\therefore$ By injectivity of $J_V$, $T^{**}x = Tx$, $\forall x$ $\therefore$ $T^{**} = T$.

Extra Credit: this is from the class (notes on web).

$J_V : V \longrightarrow V^*$ maps $y \longmapsto \langle -|y\rangle$. (We could also use the symmetric form $x \longmapsto \langle x|-\rangle$). If $\mathcal{B}$ is an orthonormal basis of $V$, its image $J_V(\mathcal{B})$ is just the dual basis

Q: If $\mathcal{B} = \{u_1, \ldots, u_n\}$, $\langle -|u_i\rangle(u_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$.

$\therefore$ $J_V(\mathcal{B}) = \{f_i \mid f_i = \langle -|u_i\rangle\}$ is the dual basis associated to $\mathcal{B}$. Then the matrix

$$\left[J_V\right]_{\mathcal{D}}^{\mathcal{B}} = \left[J_V(u_1)\big|_{\mathcal{D}}, \ldots, J_V(u_n)\big|_{\mathcal{D}}\right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & 0 & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \mathrm{Id}.$$

**This page is for extra work**

(ii)
$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ J_V^{-1}\uparrow\downarrow J_V & & \downarrow J_W \\ V^* & \xleftarrow{T'} & W^* \end{array} \qquad T^* = J_V^{-1} \cdot T' \cdot J_W$$

(iii) The adjoint satisfies: $\forall x \forall y \left[ \langle Tx|y\rangle = \langle x|T^*y\rangle \right]$.
As mentioned in class (and Berberian), $T^*$ is unique:
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Let's use the symmetric form of $T^*$:

(iv) $\langle T^{**}x|y\rangle = \langle x|T^*y\rangle = \langle T^*y|x\rangle = \langle y|Tx\rangle = \langle Tx|y\rangle$

$\therefore$ By injectivity of $J_V$, $T^{**}x = Tx$, $\forall x$ $\therefore$ $T^{**} = T$.

Extra Credit: this is from the last class (notes on web).

$J_V : V \longrightarrow V^*$ maps $y \longmapsto \langle -|y\rangle$. (We could also use the symmetric form $x \longmapsto \langle x|-\rangle$). If $\mathcal{B}$ is an orthonormal basis of $V$, its image $J_V(\mathcal{B})$ is just the dual basis

Def: If $\mathcal{B} = \{u_1, \ldots, u_n\}$, $\langle -|u_i\rangle(u_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$.

$\therefore$ $J_V(\mathcal{B}) = \{f_i \mid f_i = \langle -|u_i\rangle\}$ is the dual basis associated to $\mathcal{B}$. Then the matrix

$$\left[J_V\right]_{\mathcal{D}}^{\mathcal{B}} = \left[ J_V(u_1)\big]_{\mathcal{D}}, \ldots, J_V(u_n)\big]_{\mathcal{D}} \right] = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} = Id.$$

(5) (a) State precisely the first isomorphism theorem.

If $\varphi: V \to W$ is linear, $V/\ker(\varphi) \cong \mathrm{Im}(\varphi)$.

(b) Let $\mathcal{F}(\mathbb{R})$ be the real vector space whose elements are functions from $\mathbb{R}$ to $\mathbb{R}$, and let

$$U = \{ f \in \mathcal{F}(\mathbb{R}) \mid f(-1) = 0,\ f(0) = 0,\ \text{and } f(1) = 0 \}.$$

Show that $U$ is a subspace of $\mathcal{F}(\mathbb{R})$ and that $\mathcal{F}(\mathbb{R})/U \cong \mathbb{R}^3$.

Proof: $\underline{0} \in U$, since $\underline{0}(-1) = \underline{0}(0) = \underline{0}(1) = 0$ !

Suppose $f, g \in U$. Then $(f+g)(-1) = f(-1) + g(-1)$ $= 0 + 0$ (by hyp.) $= 0$. Also, for $\alpha \in \mathbb{R}$, $(\alpha f)(-1) = \alpha f(-1) = \alpha \cdot 0 = 0$. Similarly, $(\alpha f)(0) = \alpha f(0) = \alpha \cdot 0 = 0$; and $(\alpha f)(1) = \alpha f(1) = \alpha 0 = 0$. So $U$ is a subspace.

Define $\varphi: \mathcal{F}(\mathbb{R}) \to \mathbb{R}^3 : \varphi(f) = \big(f(-1), f(0), f(1)\big)^t$.

$\varphi$ is linear: $\varphi(f+g) = \big((f+g)(-1), (f+g)(0), (f+g)(1)\big)^t$

$= \big(f(-1), f(0), f(1)\big)^t + \big(g(-1), g(0), g(1)\big)^t$

$= \varphi(f) + \varphi(g)$.

$\varphi(\alpha f) = \big(\alpha f(-1), \alpha f(0), \alpha f(1)\big)^t$

$= \alpha \big(f(-1), f(0), f(1)\big)^t = \alpha \varphi(f)$.

$\ker(\varphi) = \{ f \mid \varphi(f) = 0 \} = U$.

claim: $\varphi$ surjective. Suppose $(a,b,c)^t \in \mathbb{R}^3$. Then pick any function $f \in \mathcal{F}(\mathbb{R})$ such that $f(-1) = a$, $f(0) = b$, $f(1) = c$ (e.g. $f(x) = \begin{cases} a & x = -1 \\ b & x = 0 \\ c & x = 1 \\ 0 & \text{else} \end{cases}$). $\therefore \varphi(f) = (a,b,c)^t$. $\therefore \mathrm{Im}(\varphi) = \mathbb{R}^3$.

$\therefore \mathcal{F}(\mathbb{R})/U \cong \mathrm{Im}(\varphi) = \mathbb{R}^3$, by the isomorphism theorem.

(6) Let $v_1, \ldots, v_n$ be a list of vectors in a vector space $V$ on a field $K$, and suppose that

$$span\{v_1, \ldots, v_n\} = V \quad \text{and} \quad v_n \in span\{v_1, \ldots, v_{n-1}\}.$$

Show that $span\{v_1, \ldots, v_{n-1}\} = V$.

Every $v \in V$ satisfies : for some $\alpha_1 \ldots \alpha_n$, $V = \sum_{i=1}^{n} \alpha_i v_i$ .

But if $v_n = \sum_{i=1}^{n-1} \beta_i v_i$ , then

$$V = \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} + \alpha_n v_n$$

$$= \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} + \alpha_n \left[ \beta_1 v_1 + \cdots + \beta_{n-1} v_{n-1} \right]$$

$$= (\alpha_1 + \alpha_n \beta_1) v_1 + (\alpha_2 + \alpha_n \beta_2) v_2 + \cdots + (\alpha_{n-1} + \alpha_n \beta_{n-1}) v_n$$

$\therefore V \in Span\{v_1 \ldots v_{n-1}\}$

$\therefore V = Span\{v_1 \ldots v_n\} \subseteq Span\{v_1, \ldots, v_{n-1}\} \subseteq V$

$\therefore Span\{v_1, \ldots, v_n\} = Span\{v_1, \ldots, v_{n-1}\}$ .

(7) (a) Show that if $S : \mathbb{R}^3 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ are two linear maps then $T \circ S$ is not injective. State precisely all results you use to show this.
(b) Give an example of linear maps $S : \mathbb{R}^3 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $S \circ T$ is injective.

Proof: Suppose $T \circ S$ injective. By a theorem in class, $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$ is also bijective. It follows that $T$ is surjective (since $T \circ S$ bijective $\Rightarrow T \circ S$ surjective. From this (by set theory) we get that $T$ is surjective (easy). Hence $T : \mathbb{R}^2 \to \mathbb{R}^3$ is surjective. This is impossible: $T(\{e_1, e_2\})$ is at most a 2 dimensional subspace of $\mathbb{R}^3$, so $T(\mathbb{R}^2) = T(\text{Span}\{e_1, e_2\})$ has dimension $\leq 2$.

(b) $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^3 \xrightarrow{S} \mathbb{R}^2$

$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} a \\ b \end{pmatrix}$

$S, T$ are clearly linear and $S \circ T = id$, which is bijective, hence injective.

(8) Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ the mapping defined by: $T(f) = f'' - 5f' - f$ for all $f \in \mathcal{P}_2$.
(a) Show that $T$ is a linear map.
(b) Give the matrix $[T]_{\mathcal{C}}^{\mathcal{C}}$, where $\mathcal{C}$ is the ordered basis $1, t, t^2$ of $\mathcal{P}_2$.
(c) Show that $T$ is an isomorphism and calculate $T^{-1}(a + bt + ct^2)$, where $T^{-1}$ is the inverse mapping and $a + bt + ct^2 \in \mathcal{P}_2$.

(a) $T(f+g) = (f+g)'' - 5(f+g)' - (f+g)$
$= f'' + g'' - 5f' - 5g' - f - g$
$= (f'' - 5f' - f) + (g'' - 5g' - g)$
$= T(f) + T(g)$

$T(\alpha f) = (\alpha f)'' - 5(\alpha f)' - (\alpha f)$
$= \alpha [f'' - 5f' - f]$
$= \alpha\, T(f)$.

(b) $[T]_{\mathcal{C}}^{\mathcal{C}} = \left[ T(1)\big|_{\mathcal{C}}, T(t)\big|_{\mathcal{C}}, T(t^2)\big|_{\mathcal{C}} \right]$

$T(1) = 1'' - 5\cdot 1' - 1 = 0 - 0 - 1 = -1$
$T(t) = t'' - 5t' - t = -5 - t$
$T(t^2) = (t^2)'' - 5(t^2)' - t^2 = 2 - 10t - t^2$

$$[T]_{\mathcal{C}}^{\mathcal{C}} = \begin{bmatrix} -1 & -5 & 2 \\ 0 & -1 & -10 \\ 0 & 0 & -1 \end{bmatrix}$$

(c) clearly $\operatorname{rank}[T]_{\mathcal{C}}^{\mathcal{C}} = 3$. $\therefore [T]_{\mathcal{C}}^{\mathcal{C}}$ is invertible, so $T$ is an isomorphism.

$$[T^{-1}] = \left[\begin{array}{ccc|ccc} -1 & -5 & 2 & 1 & 0 & 0 \\ 0 & -1 & -10 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 5 & -2 & -1 & 0 & 0 \\ 0 & 1 & 10 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\to \left[\, I \,\middle|\, \underbrace{\begin{matrix} -1 & 5 & -52 \\ 0 & -1 & 10 \\ 0 & 0 & -1 \end{matrix}}_{[T^{-1}]} \right]$$

$$[T^{-1}] = \begin{bmatrix} -1 & 5 & -52 \\ 0 & -1 & 10 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore [T^{-1}] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 & 5 & -52 \\ 0 & -1 & 10 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a + 5b - 52c \\ -b + 10c \\ -c \end{bmatrix}$$

(9) Consider the following bases: :

- The canonical basis: $\mathcal{B} : e_{11}, e_{12}, e_{21}, e_{22}$ of $M_2(\mathbb{R})$
- The basis $\mathcal{B}' : e_{11}, e_{12}, e_{21}, \epsilon$ of $M_2(\mathbb{R})$, where $\epsilon = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$
- The canonical basis $\mathcal{D} : 1, t, t^2$ of $\mathcal{P}_2$
- The basis $\mathcal{D}' : 1, t, 3t + t^2$ dof $\mathcal{P}_2$.

Suppose that the linear map $T : M_2 \to \mathcal{P}_2$ satisfies

$$[T]_{\mathcal{D}}^{\mathcal{B}} = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Calculate $T(A)$, where $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$.

(b) Find the four change of basis matrices: :

from $\mathcal{B}$ to $\mathcal{B}'$, from $\mathcal{B}'$ to $\mathcal{B}$, from $\mathcal{D}$ to $\mathcal{D}'$, and from $\mathcal{D}'$ to $\mathcal{D}$.

Give the appropriate diagram notation (as in class) and explain how you get your answers. Be very clear.

(c) Calculate $[T]_{\mathcal{D}'}^{\mathcal{B}'}$. Draw the appropriate diagram as in our lectures, and explain carefully how you arrived at your answer.

(a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. We know $M_{2\times 2} \cong \mathbb{R}^4$, $P_2 \cong \mathbb{R}^3$. $\therefore [T]: \mathbb{R}^4 \to \mathbb{R}^3$

$\therefore$ consider $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ as $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. $\therefore TA = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix}$

(b) $[id]_{\mathcal{B}}^{\mathcal{B}'}$ takes $\mathcal{B}'$-coordinates to $\mathcal{B}$ coordinates

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad (M_2(\mathbb{R}), \mathcal{B}') \xrightarrow{id} (M_2(\mathbb{R}), \mathcal{B})$

$[id]_{\mathcal{B}'}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad (M_2(\mathbb{R}), \mathcal{B}) \xrightarrow{id} (M_2(\mathbb{R}), \mathcal{B}')$

$[id]_{\mathcal{D}}^{\mathcal{D}'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad [id]_{\mathcal{D}'}^{\mathcal{D}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

$(P_2, \mathcal{D}') \xrightarrow{id} (P_2, \mathcal{D})$ $\quad (P_2, \mathcal{D}) \xrightarrow{id} (P_2, \mathcal{D}')$

(c) $(M_2(\mathbb{R}), \mathcal{B}') \to (M_2(\mathbb{R}), \mathcal{B}) \xrightarrow{[T]_{\mathcal{D}}^{\mathcal{B}}} (P_2, \mathcal{D}) \to (P_2, \mathcal{D}')$

$[id]_{\mathcal{B}}^{\mathcal{B}'}$ $\quad = \quad [T]_{\mathcal{D}'}^{\mathcal{B}'}$ $\quad [id]_{\mathcal{D}'}^{\mathcal{D}}$ $\quad$ from the class diagrams

$\therefore [T]_{\mathcal{D}'}^{\mathcal{B}'} = [id]_{\mathcal{D}}^{\mathcal{D}'} \cdot [T]_{\mathcal{D}}^{\mathcal{B}'} \cdot [id]_{\mathcal{B}}^{\mathcal{B}'}$

$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \cdot \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$

(10) Let $\langle _, _ \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$\langle (x,y,z), (x',y',z') \rangle = xx' + (x+y)(x'+y') + zz' \quad \text{for all } (x,y,z),(x',y',z') \in \mathbb{R}^3.$$

(a) Verify that $\langle _, _ \rangle$ is a real inner product on $\mathbb{R}^3$.

(b) State the Cauchy-Schwartz inequality *with respect to the precise form of the scalar product above in this question*

(c) Find an orthogonal basis of the subspace $U = \{(x,y,0) \mid x,y \in \mathbb{R}\}$ of $\mathbb{R}^3$. (Attention: this must be an orthogonal basis with respect to the scalar product above, in this question)

(a) $\langle (x,y,z) | (x,y,z) \rangle = x^2 + (x+y)^2 + z^2 \geq 0$

$\therefore$ it $= 0$ iff $x^2 = 0, (x+y)^2 = 0, z^2 = 0 \therefore x = y = z = 0$ ✓

$\langle (x,y,z) + (x'',y'',z'') \mid (x',y',z') \rangle = \langle (x+x'', y+y'', z+z'') \mid (x',y',z') \rangle =$

$(x+x'')x' + ((x+x'')+(y+y''))(x'+y') + (z+z'')z'$

$= xx' + x''x' + (x+x'')(x'+y') + (y+y'')(x'+y') + zz' + z''z'$

$= xx' + x''x + xx' + xy' + x''x' + x''y' + yx' + yy' + y''x' + y''y' + zz' + z''z'$ which is 12 terms.

$= \langle (x,y,z) | (x',y',z') \rangle + \langle (x'',y'',z'') | (x',y',z') \rangle$

$= xx' + (x+y)(x'+y') + zz' + x''x' + (x''+y'')(x'+y') + z''z'$

$= xx' + xx' + xy' + yx' + yy' + zz' + x''x + x''x' + x''y' + y''x' + y''y' + z''z'$ (the same 12 terms.

Similarly, $\langle \alpha(x,y,z) | (x',y',z') \rangle = \alpha \langle (x,y,z) | (x',y',z') \rangle$. { Symmetry is immediate since addition and multiplication are commutative.

Cauchy-Schwartz

$|\langle \vec{u} | \vec{v} \rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$. Let $\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \vec{v} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

$\therefore |xx' + (x+y)(x'+y') + zz'| \leq \sqrt{x^2 + (x+y)^2 + z^2} \sqrt{(x')^2 + (x'+y')^2 + (z')^2}$

$U = \left\{ x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mid x,y \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$V_1 \quad V_2$

$\langle (1,0,0), (0,1,0) \rangle = 0 + 1 \cdot 1 + 0 = 1 \therefore$ Not orthogonal.

$u_1 = v_1 = (1,0,0)$

$u_2 = v_2 - \text{Proj}_{u_1}(v_2) = v_2 - \frac{\langle v_2 | u_1 \rangle}{\langle u_1 | u_1 \rangle} u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{\langle (0,1,0) | (1,0,0) \rangle}{\langle (1,0,0) | (1,0,0) \rangle} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{0 + 1 \cdot 1 + 0}{1 + 1 \cdot 1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}$

$u_1 \perp u_2$, since $\langle (1,0,0) | (-1/2, 1, 0) \rangle = -\frac{1}{2} + 1 \cdot \frac{1}{2} + 0 = 0$.

(11) Let $\mathcal{L} : u_1, \ldots, u_n$ be a list of vectors in the vector space $V$ over the field $K$, and $U = Span(\mathcal{L})$ the subspace of $V$ generated by $\mathcal{L}$. Consider also a linear map $T : V \to W$, as well as the list $\mathcal{L}' : T(u_1), \ldots, T(u_n)$ of vectors from $W$.

(a) Show that if $\mathcal{L}'$ is linearly independent, then $\mathcal{L}$ is linearly independent and $U \cap \ker(T) = \{\theta_V\}$.

(b) Conversely, show that if $\mathcal{L}$ is linearly independent and $U \cap \ker(T) = \{\theta_V\}$, then $\mathcal{L}'$ is linearly independent.

(a) Suppose $\mathcal{L}'$ independent. Suppose $\sum_{i=1}^{n} \alpha_i u_i = \underline{0}$. We must show all the $\alpha_i = 0$. apply $T$ (linear). $\therefore \sum \alpha_i T(u_i) = \underline{0}$. But $\mathcal{L}'$ independent, so all $\alpha_i = 0$. $\therefore \mathcal{L}$ is independent. Now let $u \in U \cap \ker(T)$. $\therefore u \in Span(\mathcal{L})$ and $T(u) = \underline{0}$. $\therefore \exists \alpha_1, \ldots, \alpha_n$ so that $u = \sum_{i=1}^{n} \alpha_i u_i$ and $T(u) = \sum_{i=1}^{n} \alpha_i T(u_i) = \underline{0}$. But $\mathcal{L}'$ indep. $\therefore$ all $\alpha_i = 0$. so $u = \underline{0}$. $\therefore U \cap \ker(T) = \{\underline{0}\}$.

(b) Suppose $\mathcal{L}$ independent and $U \cap \ker(T) = \{\underline{0}\}$. Show $\mathcal{L}'$ indep. So suppose $\sum_{i=n} \alpha_i T(u_i) = \underline{0}$. $\therefore T(\sum_i \alpha_i u_i) = \underline{0}$. Hence $\therefore \sum \alpha_i u_i \in U \cap \ker(T) = \{\underline{0}\}$. $\therefore \sum_i \alpha_i u_i = \underline{0}$ $\therefore$ all $\alpha_i = 0$ since $\mathcal{L}$ is independent.