

MAT 2125 – Winter 2017

Quiz 3 – Solution

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March 22, 2017

QUESTION (4 pts). For which values of $a \in \mathbb{N}$ is the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^a \cos(x^{-1}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

differentiable at 0?

Solution: Let $a \in \mathbb{N}$. Then we have

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^a \cos(x^{-1})}{x} = \lim_{x \rightarrow 0} x^{a-1} \cos(x^{-1}).$$

For $a \geq 2$, we have

$$-x^{a-1} \leq x^{a-1} \cos(x^{-1}) \leq x^{a-1},$$

and $\lim_{x \rightarrow 0} x^{a-1} = 0$. Thus, by the Squeeze Theorem,

$$f'(0) = 0 \quad \text{for } a \geq 2.$$

On the other hand, if $a = 1$, then

$$f'(0) = \lim_{x \rightarrow 0} \cos(x^{-1}),$$

which does not exist. To see this consider the sequence $\{x_n\}_{n=1}^{\infty}$ given by

$$x_n = \frac{1}{n\pi}.$$

Then $x_n \rightarrow 0$, but $f(x_n) = \cos(n\pi) = (-1)^n \not\rightarrow 0$.

Thus, f is differentiable if and only if $a \geq 2$.