MAT 2125 – Winter 2017

Review – Solutions

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Overview

Final exam format

- 1 true/false question (10 statements)
- 10 other questions
- 44 total points available
- As on the midterm, you will be asked for definitions.
- Covers *all* the material in the course.

Remember that you need at least 45% on the final exam to pass the course! See

http://alistairsavage.ca/mat2125/grades/

Exam period office hours

- Monday, April 10, 1–2pm
- Wednesday, April 12, 1–2pm
- Thursday, April 13, 10:30am–12pm and 1–3pm

Review questions

Note: The choice of questions is based on requests from students. You should *not* assume that topics not covered in the review will not appear on the final exam.

Question 1. Prove that the set

$$U = \{(x,y) \in \mathbb{R}^2 : -1 < x < 2\} \subseteq \mathbb{R}^2$$

is open.

Solution: Let $(x,y) \in U$. We need to find $r > 0$ such that $B((x,y),r) \subseteq U$. (Draw a picture!) Let

$$r = \min\{x+1, 2-x\}.$$

Since $(x,y) \in U$, we have $-1 < x < 2$. Hence $x+1 > 0$ and $2-x > 0$. So $r > 0$.

Now suppose $(a,b) \in B((x,y),r)$. Then

$$|x-a| \leq \sqrt{(x-a)^2 + (y-b)^2} < r.$$

Therefore,

$$x - a \leq |x-a| < r \leq x+1.$$

Hence $a > -1$. Similarly,

$$a - x \leq |x-a| < r < 2-x.$$

Hence $a < 2$. Therefore $(a,b) \in U$. Since (a,b) was an arbitrary point of $B((x,y),r)$, we have shown that $B((x,y),r) \subseteq U$, as required.

QUESTION 2. (Exercise 8.2.10 in the notes) Prove that

$$\sum_{k=1}^{\infty} kx^{k-1}$$

converges uniformly on $[a, b]$ for all $-1 < a < b < 1$.

Solution: Fix $-1 < a < b < 1$. Let $c = \max\{|a|, |b|\} < 1$. Then, for all $x \in [a, b]$, we have

$$|kx^{k-1}| \leq kc^{k-1}.$$

Now,

$$\frac{(k+1)c^k}{kc^{k-1}} = \frac{k+1}{k}c \to c \quad \text{as } k \to \infty.$$

Since $c < 1$, the ratio test tells us that $\sum_{k=1}^{\infty} kc^{k-1}$ converges. Thus, $\sum_{k=1}^{\infty} kx^{k-1}$ converges uniformly by the Weierstrass M-test.

QUESTION 3. (Partially Exercise 8.2.2 in the notes) For $n \in \mathbb{N}$, define

$$f_n(x) = \frac{x^n}{1 + x^n}.$$

Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on intervals of the form $(-\infty, -c]$ and $[c, \infty)$ for $c > 1$ and on intervals of the form $[-d, d]$ for $0 < d < 1$. Does it converge uniformly on intervals containing the point 1?

Solution: We found the pointwise limit in Exercise 8.1.2. We found that

$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } |x| < 1, \\ 1/2 & \text{if } x = 1, \\ 1 & \text{if } |x| > 1. \end{cases}$$

(We considered the domain of the f_n to be $\mathbb{R} \setminus \{-1\}$ since $f_n(-1)$ is not defined when n is odd.)

Suppose $c > 1$. Then, for $|x| \geq c$, we have

$$|f_n(x) - 1| = \frac{1}{|1 + x^n|} \leq \frac{1}{|x|^n - 1} \leq \frac{1}{c^n - 1} \to 0 \quad \text{as } n \to \infty.$$

Thus, $f_n \rightrightarrows f$ on $(-\infty, -c]$ and $[c, \infty)$ by Exercise 8.2.9.

Now suppose $0 < d < 1$. Then, for $|x| \leq d$, we have

$$|f_n(x)| = \frac{|x|^n}{|1 + x^n|} \leq \frac{d^n}{1 - d^n} \to 0 \quad \text{as } n \to \infty.$$

Thus, $f_n \rightrightarrows f$ on $[-d, d]$ by Exercise 8.2.9.

It follows from the uniform limit theorem (Theorem 8.14) that f_n does not converge uniformly to f on any interval containing the point 1, since the pointwise limit function is discontinuous there.

QUESTION 4. Let $a, b \in \mathbb{R}$ with $a < b$. Suppose $f \colon [a, b] \to \mathbb{R}$ is twice differentiable and that it does *not* attain a global extremum at a or b. Prove that there exists some $c \in (a, b)$ such that $f''(c) = 0$.

Solution: Since f is differentiable, it is continuous by Proposition 6.5. Therefore, by the Maximum Theorem (Theorem 5.28), it attains its (global) maximum and minimum at some points u and v. By hypothesis, it $u, v \in (a, b)$. Thus, by Theorem 6.13, $f'(u) = 0 = f'(v)$. The hypotheses also imply that f is not a constant function. Hence $u \neq v$. Therefore, by the Mean Value Theorem, there exists some c between u and v such that $f''(c) = 0$.

QUESTION 5. Suppose K is a compact subset of $\mathbb{R}^n$. Show that, for any $r > 0$, K is contained in the union of a finite number of open balls of radius r.

Solution: Fix $r > 0$. Consider the collection of open balls

$$\mathcal{U} = \{B(x, r) : x \in K\}.$$

Since every $x \in K$ is contained in $B(x, r)$, the set $\mathcal{U}$ is an open cover of K. Since K is compact, there is a finite subcover of $\mathcal{U}$. In other words, K is contained in the union of a finite number of the open balls in $\mathcal{U}$ (which have radius r).

QUESTION 6. Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : 2x + 5y = 7\}.$$

Is this set closed? Is it compact?

Solution: Consider the function

$$f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = 2x + 5y.$$

Since f is a polynomial function, it is continuous. Since

$$A = f^{-1}(\{7\})$$

and $\{7\}$ is a closed subset of $\mathbb{R}$ (recall that all singletons are closed), it follows from Exercise 5.2.3 that A is closed.

The set A is not closed since it is not bounded. To see this, note that, for all $M \in \mathbb{R}$, we have

$$(M, (7 - 2M)/5) \in A \qquad \text{and} \qquad \|(M, (7 - 2M)/5)\| \geq M.$$

So A contains points of arbitrarily large norm.

QUESTION 7. For $k \in \{0, 1, 2, \dots\}$, define

$$f_k: \mathbb{R} \to \mathbb{R}, \quad f_k(x) = \frac{x^k\big(\sin(x^2) + \cos(\pi - 5x)\big)}{k!}.$$

Prove that the series $\sum_{k=0}^{\infty} f_k$ converges uniformly on any interval of the form $[a, b]$, $a < b$.

Solution: For all $x \in \mathbb{R}$, we have

$$|f_k(x)| = \frac{|x|^n}{n!}|\sin(x^2) + \cos(\pi - 5x)| \leq \frac{|x|^n}{n!}(|\sin(x^2)| + |\cos(\pi - 5x)|) \leq \frac{2|x|^n}{n!}.$$

Now suppose $a < b$, and let $c = \max\{|a|, |b|\}$. Then

$$|f_k(x)| \leq \frac{2c^n}{n!} \qquad \text{for all } x \in [a, b].$$

Since

$$\sum_{n=1}^{\infty} \frac{2c^n}{n!}$$

converges (compare to the exponential series), the series $\sum_{k=0}^{\infty} f_k$ converges uniformly by the Weierstrass M-test.

QUESTION 8. Suppose we have functions

$$f, g, h\colon [a,b] \to \mathbb{R},$$

such that

- f and h are integrable on $[a,b]$,
- $\int_a^b f = \int_a^b h$,
- $f(x) \le g(x) \le h(x)$ for all $x \in [a,b]$.

(a) Prove that g is integrable.
(b) It is necessarily true that $f(x) = g(x) = h(x)$ for all $x \in [a,b]$?

Solution:

(a) Let $\varepsilon > 0$. Since f and h are integrable, there exist partitions P' and P'' of $[a,b]$ such that

$$U(P', h) - \int_a^b h < \varepsilon/2 \qquad \text{and} \qquad \int_a^b f - L(P'', f) < \varepsilon/2.$$

Let $P = P' \cup P''$. Then we have

$$U(P,g) - L(P,g) \le U(P,h) - L(P,f) = U(P,h) - \int_a^b h + \int_a^b f - L(P,f) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore h is integrable on $[a,b]$.

(b) No, this is not necessarily true. Consider $f, g, h\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = 0, \qquad g(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0,1], \end{cases} \qquad h(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } x = 1, \\ 0 & \text{if } x \in (0,1). \end{cases}$$

QUESTION 9. Consider the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_n = \begin{cases} 5n^2 & \text{if } n \text{ is prime}, \\ -3 - \frac{4}{n} & \text{if } n \text{ is not prime}. \end{cases}$$

Find $\liminf_{n\to\infty} x_n$.

Solution: For $n \ge 3$, we have

$$\inf\{x_n, x_{n+1}, x_{n+2}, \dots\} = \begin{cases} -3 - \frac{4}{n} & \text{if } n \text{ is not prime}, \\ -3 - \frac{4}{n+1} & \text{if } n \text{ is prime}. \end{cases}$$

(Here we use the fact that, for any prime number $n \ge 3$, the number $n+1$ is even, hence not prime.) Since

$$\lim_{n\to\infty} \left(-3 - \frac{4}{n}\right) = -3 = \lim_{n\to\infty} \left(-3 - \frac{4}{n+1}\right),$$

we have

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, \dots\} = -3.$$