University of Ottawa

Department of Mathematics and Statistics

MAT 2125: Elementary Real Analysis
Professor: Alistair Savage

Midterm Test
2 March 2017

Surname ________________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	4	4	4	4	4	24
Grade							

QUESTION 1. [**4 points**] For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer. You will not lose points for incorrect answers.

____ Every bounded sequence of real numbers converges.

____ Every Cauchy sequence in $\mathbb{R}^d$ converges.

____ If a subset of $\mathbb{R}^d$ is not open, then it is closed.

____ Every bounded nonempty set of real numbers has a supremum.

____ Arbitrary unions of compact sets are compact.

____ Every monotonic sequence converges.

____ Every absolutely convergent series is convergent.

____ Every real number is a boundary point of the subset $\mathbb{Q}$ of $\mathbb{R}$.

Question 2. [**4 points**] Give an example of each of the following. You do not need to justify your answer.

(a) Series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ that are convergent, but not absolutely convergent, such that the series $\sum_{k=1}^{\infty} a_k b_k$ is absolutely convergent.

(b) A sequence $\{a_n\}_{n=1}^{\infty}$ with

$$\liminf_{n\to\infty} a_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} a_n = \infty.$$

(c) A set that is closed but not compact.

(d) Functions $f, g \colon \mathbb{R} \to \mathbb{R}$ that are not continuous (i.e. each is discontinuous at at least one point), such that $f + g$ is continuous.

QUESTION 3. Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^d$. (Do *not* assume that it is the euclidean norm.)

(a) **[1 point]** Prove that

$$\|x - y\| = \|y - x\| \quad \text{for all } x, y \in \mathbb{R}^d.$$

(b) **[2 points]** Prove that

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^d.$$

(c) **[1 point]** Suppose $\{x_n\}_{n=1}^{\infty}$ is sequence in $\mathbb{R}^d$ converging to y. Prove that $\|x_n\| \to \|y\|$ as $n \to \infty$.

Question 4.

(a) **[2 points]** Does the series

$$\sum_{n=1}^{\infty} \frac{(n+1)^2+1}{2n^2+3}$$

converge?

(b) **[2 points]** Does the series

$$\sum_{n=1}^{\infty} \frac{\sin(2n^3+5)}{n^2+1}$$

converge absolutely?

QUESTION 5.

(a) [**1 point**] Define what it means for a subset U of $\mathbb{R}^d$ to be *open*.

(b) [**3 points**] Consider the set

$$U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 > 0\} \subseteq \mathbb{R}^2.$$

Prove that U is open.

QUESTION 6.

(a) [**1 point**] Suppose $A \subseteq \mathbb{R}^d$, $A \neq \varnothing$, and $f \colon A \to \mathbb{R}^m$. Give the definition of continuity for f at a point $a \in A$. That is, complete the following sentence: "The function f is continuous at $a \in A$ if..."

(b) [**3 points**] Define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f is continuous at 0? Remember to justify your answer.

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