University of Ottawa

Department of Mathematics and Statistics

MAT 2125: Elementary Real Analysis

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Midterm Test – Solutions

2 March 2017

Surname ________________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	4	4	4	4	4	24
Grade							

QUESTION 1. [**4 points**] For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer. You will not lose points for incorrect answers.

F Every bounded sequence of real numbers converges.

T Every Cauchy sequence in $\mathbb{R}^d$ converges.

F If a subset of $\mathbb{R}^d$ is not open, then it is closed.

T Every bounded nonempty set of real numbers has a supremum.

F Arbitrary unions of compact sets are compact.

F Every monotonic sequence converges.

T Every absolutely convergent series is convergent.

T Every real number is a boundary point of the subset $\mathbb{Q}$ of $\mathbb{R}$.

QUESTION 2. [**4 points**] Give an example of each of the following. You do not need to justify your answer.

(a) Series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ that are convergent, but not absolutely convergent, such that the series $\sum_{k=1}^{\infty} a_k b_k$ is absolutely convergent.

Solution: One possible example is given by

$$a_k = b_k = \frac{(-1)^k}{k}.$$

(b) A sequence $\{a_n\}_{n=1}^{\infty}$ with

$$\liminf_{n\to\infty} a_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} a_n = \infty.$$

Solution: One possible example is

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases}$$

(c) A set that is closed but not compact.

Solution: Any set that is closed but not bounded is an example. For instance the interval $[0, \infty)$ is closed but not compact.

(d) Functions $f, g\colon \mathbb{R} \to \mathbb{R}$ that are not continuous (i.e. each is discontinuous at at least one point), such that $f + g$ is continuous.

Solution: One possible example is

$$f(x) = -g(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Then $(f + g)(x) = 0$ for all $x \in \mathbb{R}$.

QUESTION 3. Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^d$. (Do *not* assume that it is the euclidean norm.)

(a) [**1 point**] Prove that

$$\|x-y\| = \|y-x\| \quad \text{for all } x, y \in \mathbb{R}^d.$$

Solution: For $x, y \in \mathbb{R}^d$, we have

$$\|x-y\| = \|(-1)(y-x)\| = |-1| \cdot \|y-x\| = \|y-x\|.$$

(b) [**2 points**] Prove that

$$\big|\|x\| - \|y\|\big| \le \|x-y\| \quad \text{for all } x, y \in \mathbb{R}^d.$$

Solution: By the triangle inequality, we have

$$\|x\| \le \|x-y\| + \|y\| \implies \|x\| - \|y\| \le \|x-y\|.$$

Similarly, we have

$$\|y\| \le \|y-x\| + \|x\| \implies \|y\| - \|x\| \le \|y-x\| = \|x-y\|.$$

Thus

$$\big|\|x\| - \|y\|\big| = \max\{\|x\| - \|y\|, \|y\| - \|x\|\} \le \|x-y\|.$$

(c) [**1 point**] Suppose $\{x_n\}_{n=1}^{\infty}$ is sequence in $\mathbb{R}^d$ converging to y. Prove that $\|x_n\| \to \|y\|$ as $n \to \infty$.

Solution: By the above, we have

$$\big|\|x_n\| - \|y\|\big| \le \|x_n - y\| \to 0.$$

Thus $\|x_n\| \to \|y\|$ as $n \to \infty$.

QUESTION 4.

(a) [**2 points**] Does the series

$$\sum_{n=1}^{\infty} \frac{(n+1)^2+1}{2n^2+3}$$

converge?

Solution: By the arithmetic of limits, we have

$$\frac{(n+1)^2+1}{2n^2+3} = \frac{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}{2+\frac{3}{n^2}} \to \frac{1}{2} \neq 0.$$

Since the terms do not converge to zero, the series diverges.

(b) [**2 points**] Does the series

$$\sum_{n=1}^{\infty} \frac{\sin(2n^3+5)}{n^2+1}$$

converge absolutely?

Solution: We have

$$\left|\frac{\sin(2n^3+5)}{n^2+1}\right| = \frac{|\sin(2n^3+5)|}{|n^2+1|} \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}.$$

Since the series $\sum \frac{1}{n^2}$ converges, it follows from the comparison test that the given series converges absolutely.

QUESTION 5.

(a) [**1 point**] Define what it means for a subset U of $\mathbb{R}^d$ to be *open*.

Solution: The subset $U \subseteq \mathbb{R}^d$ is open if

$$\forall\, x \in U,\ \exists\, r > 0 \text{ such that } B(x,r) \subseteq U.$$

(b) [**3 points**] Consider the set

$$U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 > 0\} \subseteq \mathbb{R}^2.$$

Prove that U is open.

Solution: Let $x = (x_1, x_2) \in U$. So $x_1, x_2 > 0$. Let $r = \min\{x_1, x_2\} > 0$ and suppose $y = (y_1, y_2) \in B(x,r)$. For $i \in \{1,2\}$, we have

$$x_i - y_i \le |x_i - y_i| = \sqrt{(x_i - y_i)^2} \le \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \|x - y\| < r \le x_i.$$

Thus

$$0 < y_i \quad \text{for } i \in \{1,2\},$$

and so $y \in U$. Hence $B(x,r) \subseteq U$. Since x was arbitrary, U is open.

QUESTION 6.

(a) [**1 point**] Suppose $A \subseteq \mathbb{R}^d$, $A \neq \varnothing$, and $f \colon A \to \mathbb{R}^m$. Give the definition of continuity for f at a point $a \in A$. That is, complete the following sentence: "The function f is continuous at $a \in A$ if…"

Solution: The function f is continuous at $a \in A$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x \in A,\ |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

Equivalent definition: The function f is continuous at $a \in A$ provided that a is an isolated point of A or else that a is an accumulation point of A and

$$\lim_{x \to a} f(x) = f(a).$$

(b) [**3 points**] Define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f is continuous at 0? Remember to justify your answer.

Solution: Yes, f is continuous at 0. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/2$. Then, for all $x \in \mathbb{R}$ satisfying $0 < |x - 0| < \delta$, we have

$$|f(x) - f(0)| = \left|2x \sin\left(\frac{1}{x}\right)\right| = 2|x| \cdot \left|\sin\frac{1}{x}\right| \leq 2|x| \leq \frac{2\varepsilon}{2} = \varepsilon.$$

(Clearly, if $|x - 0| = 0$, then $x = 0$ and so $|f(x) - f(0)| = 0$.) Thus f is continuous at 0.